APPROXIMATE METHODS IN HIGH SPEED FLOW

Robert R. Burnside

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by

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DECLARATION

The undersigned hereby declares that this thesis is submitted in conformity with the regulations for the degree of Doctor of Philosophy of the University of St. Andrews.

DECLARATION

I certify that Mr. R. R. Burnside has fulfilled the conditions of the Ordinance and Regulations for the presentation of the following thesis.

Research Supervisor

PERSONAL FOREWORD

From 1954-1958, the writer was a student at the Royal College of Science and Technology and in 1958 was awarded the A. R. C. S. T. with first class honours in Applied Mathematics. In October of that year, he was accepted as a research student of St. Salvator's College, University of St. Andrews and from that time till June 1961 worked on the subject matter of this thesis under the supervision of Dr. A. G. Mackie. In September 1961, he was accepted as a Lecturer in Mathematics by the Senate of the University of Toronto and whilst there completed the composition of this thesis.

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I wish to express my most sincere appreciation to Dr. A. G. Mackie for his willing and patient supervision of this thesis. In addition, my thanks are due to the Department of Scientific and Industrial Research for the Research Studentship which I held whilst engaged on this work, and also, to Miss J. Williams of the University of Toronto for her painstaking attention to detail in the typing of the manuscript.

PREFACE

The thesis is presented in two volumes, the first of which consists of the actual material covered whilst in the second are grouped together the relevant figures, graphs, numerical data and appendices which are referred to in the first volume.

CONTENTS

CHAPTER I

INTRODUCTION

In many problems arising in the theory of compressible flow, the equations characterising the solution of the system are so intractable that recourse must be made to some approximate method which allows the essential features of the flow to be preserved, whilst to some degree, simplifying the mathematics. It is with certain methods of this type that this thesis is concerned.

In the subsequent work, we shall assume that the effects due to viscosity and heat conduction are so small as to be negligible. These assumptions may be shown to be largely valid except in those domains of the flow-field where the modified system of equations predicts regions in which the solution is in general multivalued. In the modified system, however, such 'regions' are avoided by the introduction of mathematical discontinuities and, assuming that the jump conditions across them can be determined, are sufficient to provide single-valued solutions valid everywhere, except at the discontinuity.

The methods to be presented are formulated in the plane consisting of one space variable and one time variable.

§1. THE EQUATIONS OF GAS DYNAMICS

a. The Thermodynamic Equations

The fluid is postulated to be homogeneous and permanent and such that the laws of Boyle and Gay-Lussac are satisfied; that is, it is an ideal or perfect fluid. We shall also assume that the specific internal energy is simply proportional to the absolute temperature, thus it is in addition a polytropic fluid.

Let $p(x,t)$, $\rho(x,t)$ and $S(x,t)$ denote the specific pressure, density and entropy at any point of the fluid. These functions are assumed to be continuous and differentiable as required. The condition that the fluid is ideal implies that the equation of state is

$$p = \rho R T , \tag{1.1.1}$$

where T is the absolute temperature and R is the quotient of the universal gas constant and the effective molecular weight of the particular fluid.

The assumption that the fluid is also polytropic leads in conjunction with equation (1.1.1) to the entropic equation of state, namely

$$p(\rho, S) = A\rho^{\gamma}, \tag{1.1.2}$$

in which the coefficient A depends on the entropy, S, and γ, the adiabatic index, is the ratio of the specific heats at constant pressure and volume, c_p and c_v respectively, of the given fluid.

For the class of fluids under consideration here, equation (1.1.2) may be shown to be given explicitly by the relation (Courant and Friedrichs, 1948)

$$\frac{p}{p_o} = \left(\frac{\rho}{\rho_o}\right)^{\gamma} \exp \frac{S - S_o}{c_v} \tag{1.1.3}$$

where the suffix 'o' signifies that the particular quantities are measured in some standard state.

If there is no external source of heat, then as the effects of viscosity and heat conduction are neglected, no heat is added to the thermodynamic system and thus changes of state are adiabatic. From the first and second laws of thermodynamics, such changes of state imply that the specific entropy of a moving particle remains constant; that is, for any particle of an adiabatic system

$$\frac{DS}{Dt} = 0, \tag{1.1.4}$$

where $\frac{D}{Dt}$ denotes the full derivative following the particle. If $u(x, t)$

denotes the velocity of a particle of the fluid, then

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} .$$

Relation (1.1.4) does not hold, however, across discontinuities of the type referred to above where, for an adequate description of the flow, the effects of viscosity and heat conduction must now be included.

The particle paths of the fluid are determined by the appropriate solution of the equation

$$\frac{dx}{dt} = u(x, t) ,$$

and on those curves we have the condition (1.1.4) holding.

In all real fluids, the pressure increases as the density, and vice versa, assuming that the entropy is constant. Accordingly, from equation (1.1.2) this implies that

$$\{\frac{\partial p(\rho . S)}{\partial \rho}\}_S > 0$$

for all states in the flow, except the limiting case $\rho = o$, for which $(\frac{\partial p}{\partial \rho})_S = 0$.

A positive quantity, c, having the dimensions of velocity, may therefore be defined, such that

$$c^2 = \frac{\partial p}{\partial \rho} = \{\frac{\partial p(\rho . S)}{\partial \rho}\}_S = \frac{\gamma p}{\rho} \, , \tag{1.1.5}$$

from equation (1.1.3). This quantity, c, can be identified as the local speed of sound, a term which will be justified later.

For any value of S, the function $p(\rho . S)$ is generally either convex downward or linear, that is

$$\frac{\partial^2 p}{\partial \rho^2} \geq 0 \, .$$

The adiabatic index, γ, must therefore satisfy the inequality

$$\gamma \geq 1 \, ,$$

for the fluids with which we are concerned.

b. Euler's Equations

The equations formulated here are based on the principles of Newtonian mechanics and express the law of conservation of mass and the equation of motion.

(i) Conservation of Mass

Consider my fixed control surface, S, enclosing a volume, V, in the fluid, in which there is a point x_i with velocity u_i. Assuming that there are no sources or sinks within S, it is permissable to equate the rate of increase of mass in V to the inflow into this volume;

that is,

$$\frac{\partial}{\partial t} \int_V \rho \, dV = - \int_S \rho u_i \, dS_i \, .$$

On applying Green's theorem to the surface integral and using the condition that the choice of control surface is arbitrary, the following result is obtained, which is valid at any point in the fluid.

$$\frac{\partial \rho}{\partial t} + \operatorname{div} (\rho u_i) = 0 \, . \tag{1.1.6}$$

For one dimensional unsteady motion, this result reduces to

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0.$$

(ii) Equation of Motion

Consider a volume V bounded by a surface S, of similar properties to that above, but of infinitesimally small dimensions. Since no viscous forces are present, the stress tensor reduces to the simple form

$$- p n_i \, ,$$

where n_i is the outward drawn normal to the surface, S, at the point x_i.

If no external forces act on the system, then Newton's 2nd law of motion yields,

$$\rho V \frac{Du_i}{Dt} = - \int_S p n_i \, dS \, .$$

By applying Green's Theorem to the surface integral, expanding the resulting volume integral and then taking the limits as V tends to zero, the equation of motion is seen to be given by,

$$\rho \frac{Du_i}{Dt} = - \nabla p \, . \tag{1.1.7}$$

For one dimensional, unsteady flows, this relation reduces to

$$\rho \frac{Du}{Dt} = - \frac{\partial p}{\partial x} \, .$$

It is convenient to group the equations characterising the one dimensional, unsteady motion of a perfect, polytropic fluid and quote them in a form in which the density, ρ, is eliminated in favour of the velocity of sound, c, and the entropy, S, by means of equations (1.1.3), (1.1.4), (1.1.5).

a. Conservation of Mass:

$$\frac{c_t}{\gamma - 1} + u \frac{c_x}{\gamma - 1} + c \frac{u_x}{2} = 0 \, .$$

b. Equation of Motion:

$$\frac{u_t}{2} + c\,\frac{c_x}{\gamma-1} + u\,\frac{u_x}{2} = \frac{c^2}{2\gamma(\gamma-1)c_v}\,\frac{\partial S}{\partial x}\,.$$

(1.1.8)

c. Adiabatic Changes of State:

$$S_t + uS_x = 0\,.$$

d. Polytropic Equation of State:

$$\left(\frac{p}{p_o}\right) = \left(\frac{c}{c_o}\right)^{\frac{2\gamma}{\gamma-1}} e^{\frac{S-S_o}{\gamma-1\;c_v}}$$

The equations a, b, c form a totally hyperbolic system of the 3rd order in the three unknowns u, c and S, the solution of which may be expected to be uniquely determined when the boundary conditions appropriate to a given problem are applied. The solution for the complete system is obtained by using the polytopic equation of state to determine the pressure, p, when c and S have been found from the dynamical equations.

c. <u>The Characteristics and Compatibility Conditions of the System (1.1.8)</u>

If new dependent variables $\alpha(x,t)$, $\beta(x,t)$ are defined by

$$\alpha = \frac{u}{2} + \frac{c}{\gamma-1}$$

(1.1.9)

$$\beta = -\frac{u}{2} + \frac{c}{\gamma-1}\;,$$

then on addition and subtraction of equations (1.1.8) a, b, we obtain

$$\alpha_t + (u+c)\,\alpha_x = \frac{c^2}{2\gamma\,.\,\gamma-1\;c_v}\;\frac{\partial S}{\partial x}\;.$$

(1.1.10)

$$\beta_t + (u-c)\,\beta_x = -\frac{c^2}{2\gamma\,.\,\gamma-1\;c_v}\;\frac{\partial S}{\partial x}\;.$$

On inspection of equations (1.1.10) and (1.1.8) c, it may be seen that the flow-field is covered by a network of three sets of real characteristics, which by convention are referred to as the C^+, C^- and C^o characteristics, determined respectively by the relations,

$$\text{a.}\quad C^+\,.\;\frac{dx}{dt} = u + c\;.$$

$$\text{(1.1.11)}\quad \text{b.}\quad C^-\,.\;\frac{dx}{dt} = u - c\;.$$

$$\text{c.}\quad C^o\,.\;\frac{dx}{dt} = u\;.$$

The quantity c is by definition positive and therefore the relative orientation of the characteristics through a given point, $P(x, t)$, in the field is invariant. For if the direction of increasing x is from left to right, then the characteristic directions of C^+, C^o and C^- are always in the same order, namely anticlockwise. From the classical theory of hyperbolic equations, discontinuities in the first derivatives of the flow variables are propogated into the fluid with the local velocity of sound along the forward-facing C^+ characteristics and the backward-facing C^- characteristics. These characteristics may thus be regarded as the paths of wave fronts on which small disturbances are transmitted into the fluid. Acoustic phenomena are manifested in this way which justifies the term 'velocity of sound' for c.

The characteristics afford a means of 'patching' together analytically different solutions of a hyperbolic system and form the basis of a numerical technique by which the solution of any well-formulated problem of this system may be obtained to any required accuracy. As we are concerned here with approximate methods based on an analytic determination of the flow, no further details of this technique are given.

For the first derivatives of the flow variables to remain finite on the characteristics, certain compatability relations must hold on these curves. From the equations (1.1.10) and (1.1.8) c, it is evident that the necessary conditions are:

$$\text{a. on } \frac{dx}{dt} = u + c, \quad D_+\alpha = \frac{c^2}{2\gamma \cdot \gamma-1\ c_v} \frac{\partial S}{\partial x} = \frac{c}{2\gamma(\gamma-1)c_v} D_+ S$$

$$(1.1.12) \quad \text{b. on } \frac{dx}{dt} = u - c, \quad D_-\beta = -\frac{c^2}{2\gamma \cdot \gamma-1\ c_v} \frac{\partial S}{\partial x} = +\frac{c}{2\gamma(\gamma-1)c_v} D_- S$$

$$\text{c. on } \frac{dx}{dt} = u, \qquad D_o S = 0,$$

where D_+, D_- and D_o are operators denoting differentiation with respect to time along the C^+, C^- and C^o characteristics respectively, that is,

$$D_+ = \frac{\partial}{\partial t} + (u+c)\frac{\partial}{\partial x} \ ,$$

$$D_- = \frac{\partial}{\partial t} + (u-c)\frac{\partial}{\partial x} \ ,$$

$$D_o = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} \ .$$

The constant value for the entropy is determined by the initial conditions in the particular problem.

This alternative formulation of the governing equations, together with the equation of state, is often preferable to that given by the system (1.1.8), as it provides a natural and elegant way of describing many flows.

§2. ISENTROPIC SIMPLE WAVES

a. Basic Concepts

If the flow is everywhere isentropic, then the characteristic relations 1.1.12, integrate to yield the simple results,

$$\text{a.} \quad \alpha = \alpha_o \ , \quad \text{on} \ \frac{dx}{dt} = u + c \ ,$$

$$\text{(1.2.1)} \qquad \text{b.} \quad \beta = \beta_o \ , \quad \text{on} \ \frac{dx}{dt} = u - c \ ,$$

c. S is constant everywhere,

where the constants of integration α_o, β_o are supplied by initial data.

If the roles of the dependent and independent variables are interchanged, the isentropic system reduces to a second order, linear partial differential equation of the Euler-Poisson-Darboux type in either of the physical variables x, t. Points in the physical plane have a one-to-one correspondence with those of the characteristic plane so long as the Jacobian of the transformation, $\frac{\partial (\alpha, \beta)}{\partial (x, t)}$, is non-singular. In general, any finite region of the (x, t)-plane is mapped onto a finite area in the (α, β)-plane. The extreme degenerate case is when the flow is uniform: α = constant, β = constant. The entire physical plane in this instance corresponds to a single point of the characteristic plane.

A case of particular importance occurs when an element of area in the physical plane is mapped onto a single arc in the characteristic plane. This situation occurs only when the Jacobian of the transformation, $\frac{\partial (\alpha, \beta)}{\partial (x, t)}$, vanishes at all points. Three cases can be distinguished:

(i) α = constant, β = constant,

(ii) α = constant, with β variable,

(iii) β = constant, with α variable.

In case (i), the solution corresponds to that of uniform flow including the special case $\alpha = -\beta$ = constant which corresponds to a vacuum, that is the density and pressure are zero. Clearly, this type of flow is represented in the characteristic plane by an 'arc' consisting of one point.

Cases (ii), (iii) introduce the concept of the simple wave. To any theorem derived for α = constant there is a dual one for β = constant. Accordingly, two kinds of simple wave may be distinguished: when β is constant, the simple wave is described as 'forward-facing', whilst when α is constant the simple wave is said to be 'backward-facing'. A forward-facing simple wave is then mapped onto the arc β constant in the characteristic plane and a backward-facing wave onto α constant. For a forward-facing simple wave, equation (1.1.10) has solutions of the form

$$\alpha = \alpha\left(\frac{x-a}{t-b}\right),$$

where a, b are constants. The C^+ characteristics on which α is constant are therefore straight and the cross-characteristics, C^-, are in general curved. For a backward-facing wave, the situation is reversed.

Each basic type of simple wave is further subdivided into two distinct groups. If the pressure and density of a fluid particle decrease with time increasing, then the wave is called a rarefaction or expansion wave. Conversely, if the pressure and density increase, then the wave is a compression or condensation wave.

Using the thermodynamic properties derived in section 1a, it can be shown that the velocity of propagation of sound waves, on the C^+, C^- characteristics in a forward-facing and backward-facing simple wave respectively, changes as the fluid velocity; that is

$$d\frac{(u+c)}{du} > 0 \,, \quad \text{in a forward-facing simple wave.}$$

$$d\frac{(u-c)}{du} > 0 \,, \quad \text{in a backward-facing simple wave.}$$

Taken in conjunction with the above remarks on rarefaction and compression waves, these inequalities indicate that the straight characteristics in rarefaction waves diverge, whilst those of the compression wave converge.

Since a physically possible flow is represented only by portions of the (x, t)-plane in which the straight characteristics do not cross each other, the following basic difference between rarefaction and compression waves emerges. For given initial conditions, a rarefaction wave can extend for an infinitely long time, whilst the compression wave is restricted to a finite interval of time.

The four types of simple waves are illustrated in Figure 1.

An important theorem concerning simple waves is that the flow in any domain adjacent to a region of uniform flow is a simple wave. To prove this we note that if the region '1' in the (x, t) plane is bounded on the right by region '2' of constant flow, as in Figure 2, then the characteristics in this latter region are straight. If the boundary between the two regions is a C^+ characteristic, then the lines C^+ in one region do not enter the other. On the other hand, the characteristics C^- pass continuously from one region to the other carrying the constant value β_o. Thus β is constant throughout region '1' so that the flow there is a forward-facing simple wave.

b. Properties of Simple Waves

Since the properties of backward-facing and forward-facing simple waves are derived in similar manners, we shall deduce only those pertaining to the latter type of wave.

It is customary to refer to the characteristic variables (α, β), which are constant on the respective characteristics, as the Riemann Invariants. Thus, for a forward-facing simple wave, the Riemann invariant β has a constant value throughout the flow-field.

In the following work, a suffix 'o' is used to denote conditions taken in a uniform flow region, adjacent to a simple wave, in which the fluid velocity, u_o, is positive.

Since a forward-facing wave is characterised by the condition that the Riemann invariant, β, has the value β_o, the compatability condition on the C^- characteristics yields the result

$$-\frac{u}{2}+\frac{c}{\gamma-1} = -\frac{u_o}{2}+\frac{c_o}{\gamma-1} ,$$

(1.1.2)

that is $$(c - c_o) = \frac{\gamma-1}{2}(u - u_o) .$$

The quantities p, ρ can now be obtained at functions of the velocity difference, $u - u_o$, from the isentropic form of the equation of state (1.1.8)d and equation(1.1.5), whence

$$\rho = \rho_o\left[1+\frac{\gamma-1}{2}\,\frac{u-u_o}{c_o}\right]^{\frac{2\gamma}{\gamma-1}}$$

(1.2.3)

$$\rho = \rho_o\left[1+\frac{\gamma-1}{2}\,\frac{u-u_o}{c_o}\right]^{\frac{2}{\gamma-1}}$$

To determine the simple wave completely, it is necessary, therefore, to have knowledge of only one of the variables u, c, p or ρ.

We now derive the basic analytic expressions for simple rarefaction and compression waves, the latter leading naturally into a discussion of shock phenomena.

Rarefaction Waves. It is easier to discuss these questions physically by relating them to the motion of a piston bounding a semi-infinite column of fluid. If initially the fluid and piston are moving with uniform velocity, u_o, then what may be said of the resulting flow if after a finite time the motion of the piston is retarded ?

Let x denote the distance of a particle of the fluid measured from the time $t = o$, when the piston acceleration first becomes negative.

A suffix 'p' is used to denote parameters taken on the piston path. The motion of the piston is given by

$$x_p = X_1(t_p) ,$$

where X_1 is a continuous and differentiable monotonic decreasing function, such that

(i) $X_1'(0) = u_o$

(ii) $X_1''(t_p) \leq 0 , \quad t_p \geq 0$,

where dashed symbols denote differentiation w.r.t t_p.

Conditions (i), (ii) ensure that the transition of the piston motion from a uniform state to a retarded state is continuous.

The flow-field in the (x, t)-plane is as illustrated in Figure 3.

The flow resulting from the motion of the piston is confined to the domain

$$x \leq (u_o + c_o)t \,.$$

This region is adjacent to one of uniform flow and the field enters from the right. The flow in it is therefore a forward-facing simple rarefaction wave, with Riemann invariant β_o and the C^+ characteristics are straight. On the piston, the fluid velocity is

$$u = X_1'(t_p) \,.$$

In general the fluid velocity, u, and the local speed of sound, c, found from the compatability condition on the C^+ characteristics, the Riemann invariant, β_o, and the above result for the velocity of the fluid on the piston, are

$$(1.2.4) \qquad \begin{aligned} u &= X_1'(t_p) \,, \\ c &= c_o + \frac{\gamma-1}{2}\,[X_1'(t_p) - u_o] \,. \end{aligned}$$

The pressure, p, and the density, ρ, are then obtained in terms of the given data, $X_1(t_p)$, u_o and c_o, from the relations (1.2.3).

Since the velocity of the forward-facing sound waves changes in the same sense as the fluid velocity, the C^+ characteristics diverge from the piston path, as indicated in Figure 3.

That the foregoing discussion must be modified if the final piston velocity, u_f, exceeds a certain limit, is evident from relation (1.2.3). The maximum expansion that can be attained corresponds to a wave which is completed by vacuum conditions; $p = o$, $\rho = o$, that is when

$$u_f = u_o - \frac{2}{\gamma-1} c_o = -2\beta_o .$$

The simple wave, called a completed simple wave, thins the fluid down to zero density, pressure and consequently zero sound speed. If, however, $u_f > -2\beta_o$, then the wave is completed on the characteristic through the point on the piston path for which $u = u_f$. Between the tail of the incomplete rarefaction wave and the piston path, there is a region of uniform flow in which $p = p_f$, $\rho = \rho_f$, $u = u_f$. The final wave feature occurs if $u_f < -2\beta_o$. The rarefaction wave is completed on the characteristic on which $\rho = o$ and between the tail of the wave and the piston path is a zone of cavitation, that is, a vacuum.

The solution for the simple rarefaction wave is completed by obtaining the equations of the characteristics as functions of the independent variables, (x, t).

In terms of the Riemann invariants, α and β_o, the particle velocity, u, and the local speed of sound may be expressed thus

$$\begin{aligned} u &= \alpha - \beta_o \\ c &= \frac{\gamma-1}{2}(\alpha + \beta_o) . \end{aligned} \tag{1.2.5}$$

On the piston curve, we have also from equations (1.2.4)

$$u = X_1'(t_p)$$

$$c = c_o + \frac{\gamma - 1}{2}(X_1'(t_p) - u_o)\,,$$

so that we may put

$$(1.2.6) \qquad x = X(\alpha)\,, \quad t = T(\alpha)$$

on the piston curve, where X and T are known functions.

The equation of the C^+ characteristics is then

$$(1.2.7) \qquad x - X(a) = \left(\frac{\gamma+1}{2}\alpha - \frac{3-\gamma}{2}\beta_o\right)(t - T(\alpha))\,,$$

and that for the C^- characteristics from (1.1.11) now becomes

$$(1.2.8\text{ a}) \qquad \frac{dx}{dt} = \frac{3-\gamma}{2}\alpha - \frac{\gamma+1}{2}\beta_o\,.$$

Equation (1.2.7) provides a relation for x in terms of t and α which is valid throughout the simple wave domain. We may therefore eliminate either x or t in (1.2.8) using (1.2.7) to obtain a linear, first order differential equation in either x or t. In the independent variable, t, this equation is

(1.2.8 b)

$$\frac{dt}{d\alpha}+\frac{\gamma+1}{2(\gamma-1)}\frac{t}{(\alpha+\beta_o)}=\frac{\gamma+1}{2(\gamma-1)}\frac{T}{(\alpha+\beta_o)}-\frac{X'}{(\gamma-1)(\alpha+\beta_o)}+\left\{\frac{(\gamma+1)\alpha-(3-\gamma)\beta_o}{2(\gamma-1)(\alpha+\beta_o)}\right\}T'',$$

and for given initial conditions, the definite integral is easily obtained.

The equation of the particle paths, (1.1.11), in terms of α and β_o is

(1.2.9 a)
$$\frac{dx}{dt}=\alpha-\beta_o\ ,$$

and may be reduced to a first order, linear differential equation in either x or t in a manner similar to that above. In terms of t, this equation is

(1.2.9 b)

$$\frac{dt}{d\alpha}+\frac{\gamma+1}{(\gamma-1)}\frac{t}{(\alpha+\beta_o)}=\frac{\gamma+1}{(\gamma-1)}\frac{T}{(\alpha+\beta_o)}-\frac{2}{(\gamma-1)}\frac{X'}{(\alpha+\beta_o)}+\left\{\frac{(\gamma+1)\alpha-(3-\gamma)\beta_o}{(\gamma-1)(\alpha+\beta_o)}\right\}T'.$$

A rarefaction wave of considerable importance to the later work is formed when the retardation of the piston takes place instantaneously. The family of C^+ characteristics degenerates to a pencil of lines through the origin, taken where the piston motion changes, and the resulting flow is known as a 'point-centred' simple rarefaction wave. The flow parameters may be obtained in a manner similar to that given above for the general case, with the simplification that the function $X_1(t_p)$ is now of the form

$$x_p = u_1 t_p ,$$

where u_1 is the constant velocity of the piston for values of time greater than zero, clearly $u_1 < u_o$. The point-centred simple wave, illustrated in Figure 4, affords an example of an initial discontinuity which is immediately resolved into continuous flow.

Compression Waves. If the piston is accelerated into the fluid instead of being withdrawn, then a simple compression wave originates at the piston. The above formulae and comments pertaining to the rarefaction wave are still largely applicable except that now the density, pressure and particle velocity increase with time. Since the piston velocity is an increasing function of time, the C^+ characteristics will eventually intersect and, as shown in Figure 5, will form an envelope on which the solutions of the governing equations are multi-valued and thus physically inadmissable.

The representation of the envelope is determined as follows. The fluid is assumed to be at rest initially and consequently c_o now denotes the velocity of sound under stagnation conditions. The piston path is given by

$$x_p = X(t_p) ,$$

where X is a continuous and differentiable monotonic increasing function in the interval $0 \le t \le t^{x}$, satisfying the conditions

$$(1.2.10) \qquad \begin{aligned} &\text{(i)} && X(0) = 0 = \dot{X}(0) \\ &\text{(ii)} && \ddot{X}(\lambda) > 0, \quad \text{for } \lambda > 0 \\ &\text{(iii)} && \ddot{X}(t) = 0, \quad \text{for } t \ge t^{x}, \end{aligned}$$

where t^{x} is chosen such that only one envelope is formed by the C^{+} characteristics generated by the piston motion and λ is a parameter denoting the time at which the C^{+} characteristic through the point (x, t) started at the piston.

From (1.2.7), the resulting simple wave may be represented by

$$(1.2.11) \qquad x = X(\lambda) + [\frac{\gamma+1}{2}\,\alpha(\lambda) - \frac{3-\gamma}{2}\,\beta_o](t-\lambda).$$

On substituting for $\alpha(\lambda)$ in terms of $X(\lambda)$ and c_o from equation (1.2.4) with $u_o = 0$, the above equation may be written as

$$(1.2.12) \qquad x = X(\lambda) + [\frac{\gamma+1}{2}\,X'(\lambda) + c_o](t-\lambda).$$

The envelope of this family of characteristics is determined by the condition that the derivative of x w.r.t. λ vanishes, so that

$$(1.2.13) \qquad t(\lambda) = \frac{\frac{\gamma-1}{\gamma+1}\,X'(\lambda) + \frac{2}{\gamma+1}\,c_o}{X''(\lambda)} + \lambda$$

and $$x(\lambda) = \frac{\left[\frac{\gamma+1}{2} X'(\lambda) + c_o\right]\left[\frac{\gamma-1}{\gamma+1} X'(\lambda) + \frac{2}{\gamma+1} c_o\right]}{X''(\lambda)} + X(\lambda)$$

are the parametric equations of the envelope.

Since $X(\lambda)$ and the derivatives w.r.t λ are positive, it is evident that for $\lambda > 0$, the envelope is formed in the flow region $x > X(\lambda)$. The minimum value for t on the envelope occurs in general when $\frac{dt}{d\lambda} = 0$, that is when λ satisfies the equation

$$(1.2.14) \qquad \frac{\left(\frac{\gamma-1}{\gamma+1} X'(\lambda) + \frac{2}{\gamma+1} c_o\right)}{X''(\lambda)} = \lambda + \frac{2\gamma}{\gamma+1} \frac{X''(\lambda)}{X'''(\lambda)} .$$

Since $\frac{dx}{d\lambda}$ is also zero at this point, the envelope has a cusp at the minimum point.

When $\overset{\prime\prime}{X}(o) > 0$, the initial point of the envelope, $N(x_n, t_n)$ is obtained by putting $\lambda = 0$ in (1.2.13), that is

$$(1.2.15) \qquad x_n = \frac{2\, c_o^2}{\gamma+1 \,.\, X''(0)} \quad ; \quad t_n = \frac{2\, c_o}{\gamma+1 \,.\, X''(0)} .$$

The initial point of the envelope therefore lies on the first C^+ characteristic of the simple compression wave,

$$x = c_o t ,$$

as indicated in Figure 5.

When $\ddot{X}(0) = 0$, then to obtain the coordinates of the initial point of the envelope, equation (1.2.14) must be solved for λ, which is then substituted into (1.2.13) to yield the required coordinates. It is also important to note that the envelope is formed in the simple wave region.

When $\ddot{X}(0) < 0$, then the piston is retarded and there is no point of the envelope in the domain $x > X(\lambda)$ corresponding to the interior of the (x,t) domain of the flow, as is expected.

§3. SHOCK WAVES

The occurrence of multivalued solutions for the flow variables in the domain of the envelope of the C^+ characteristics in the simple compression wave necessitates that we reassess the physical assumptions on which equations (1.1.8) are based. These assumptions are tenable only when the velocity and pressure gradients are small. In the neighbourhood of the envelope, however, this is no longer so. The effects due to viscosity and heat conduction must now be included for an adequate description of the flow in this region. In this domain, the thermodynamic changes will be irreversible and consequently the entropy of a particle will increase. It has been observed in fluids that such effects are confined to very narrow zones, of the order of the mean free path of a particle, whilst outside the zone the flow is such as may be described by an adiabatic reversible process. This fact suggests that as a mathematical idealisation of the flow structure, these zones may be replaced in the (x, t)-plane by curves, termed 'shock loci', across which the dependent variables change discontinuously and the thermodynamic process is irreversible. The flow on either side may then be obtained from the system of equations (1.1.8). It should be noted, however, that as the conditions of inviscid flow are violated, flow patterns which include shock fronts cannot be considered as discontinuous solutions of the differential equations for a perfect fluid. They are to be regarded rather as asymptotic solutions of the equations for viscous flows when the viscosity tends towards zero.

In the compression wave problem a shock curve would be introduced into the flow-field, beginning at the initial point of the envelope and continuing so that it cuts each C^+ characteristic before the latter intersects with the characteristics of the same family. It is important to note that shock loci are not characteristics.

This idealisation will be fruitful provided that the jump conditions across the discontinuity are determinate. We now determine these conditions, known as the Rankine-Hugoniot shock relations, by applying the principles of conservation of mass, momentum and energy together with the conservation or increase of entropy to a column of fluid.

The two regions separated by the shock, termed 'ahead' and 'behind', are specified by assuming that the particles pass through the shock from ahead to behind. We shall always assume that the region ahead of the shock is the one in which x may increase without limit. Quantities in those regions will be denoted respectively by the suffices '0' and '1'.

We assume that the column of fluid at time t covers the domain $a_0(t) < x < a_1(t)$, where $a_0(t)$, $a_1(t)$ denote the positions of particles at the ends of the columns, and that the shock locus is specified by $\xi(t)$, where $a_0(t) < \xi(t) < a_1(t)$.

On applying the above conservation principles to this column, we then obtain the following relations.

a. Conservation of Mass: $$\frac{d}{dt}\int_{a_0(t)}^{a_1(t)} \rho(x,t)\,dx = 0 .$$

b. Conservation of Momentum: $$\frac{d}{dt}\int_{a_0(t)}^{a_1(t)} \rho u\,dx = p(a_0,t) - p(a_1,t) .$$

(1.3.1)

c. Conservation of Energy:
$$\frac{d}{dt}\int_{a_0(t)}^{a_1(t)} \rho\left(\frac{1}{2}u^2 + e\right) dx = p(a_0,t)\,u(a_0,t) - p(a_1,t)\,u(a_1,t) ,$$

in which e denotes the specific internal energy of the fluid.

d. Increase or Conservation of Entropy:

$$\frac{d}{dt}\int_{a_0(t)}^{a_1(t)} \rho\, S\, dx \geq 0 \,.$$

Of the above relations, equation (a) is obvious while equations (b) and (c) express the respective assumptions that the only forces acting are pressure forces and that the total gain in energy is due entirely to the action of these forces. Relation (d) states that the column either maintains or gains entropy.

All the preceding integrals are of the form

$$J = \int_{a_0(t)}^{a_1(t)} \psi(x,t)\, du \quad ;$$

where the integrand ψ is discontinuous at the interior point $x = \xi(t)$ of the interval of integration.

Differentiating w. r. t time t, we then obtain

$$\frac{dJ}{dt} = \frac{d}{dt}\int_{a_0(t)}^{\xi(t)} \psi(x,t)\, du + \frac{d}{dt}\int_{\xi(t)}^{a_1(t)} \psi(x,t)\, du \;.$$

$$= \int_{a_0(t)}^{a_1(t)} \frac{\partial \psi(x,t)}{\partial t}\, dx + \psi_0 \dot{\xi}(t) - \psi(a_0,t)\, u(a_0,t) + \psi(a_1,t)\, u(a_1,t) - \psi_1 \dot{\xi}(t) \,,$$

where $u(a_0, t)$, $u(a_1, t)$ denote respectively the velocities of the particles at the ends of the column, and $\dot{\xi}(t)$ is the velocity of the shock. ψ_0, ψ_1 are the limit of $\psi(x, t)$ as $x \to \xi$ from above and below respectively.

We now perform the limiting process, allowing the length of the column of fluid to tend to zero, that is

$$\lim_{(a_1 - a_0) \to 0} \frac{dJ}{dt} = \psi(u_1 - \dot{\xi}) - \psi_0(u_0 - \dot{\xi}) .$$

Thus we derive from relations (1.3.1), the four basic jump conditions relating quantities on either side of the shock:

a. Conservation of Mass: $\rho_1(u_1 - \dot{\xi}) = \rho_0(u_0 - \dot{\xi})$,

b. Conservation of Momentum:

$$\rho_1(u_1 - \dot{\xi})u_1 - \rho_0(u_0 - \dot{\xi})u_0 = p_0 - p_1 ,$$

(1.3.2)

c. Conservation of Energy:

$$\rho_1\left(\frac{1}{2}u_1^2 + \frac{p_1}{(\gamma-1)\rho_1}\right)(u_1 - \dot{\xi}) - \rho_0\left(\frac{1}{2}u_0^2 + \frac{p_0}{(\gamma-1)\rho_0}\right)(u_0 - \dot{\xi})$$

$$= p_0 u_0 - p_1 u_1 ,$$

d. Conservation or Increase of Entropy:

$$\rho_1(u_1 - \dot{\xi})S_1 \geq \rho_0(u_0 - \dot{\xi})S_0 .$$

In equation (c), we have made use of the substitution

$$e = \frac{p}{\gamma-1}\rho \quad ,$$

which is valid for a perfect fluid.

If conditions on one side of the shock are given together with the shock velocity, $\dot{\xi}$, then the above equations may be solved uniquely for the four unknowns u, p, ρ and S on the 'other' side.

Using relation (a) above we may write (d) in the form

$$S_1 \geq S_0 \quad , \tag{1.3.3}$$

that is, the entropy of a particle does not decrease on passing through a shock.

To derive some properties of shock flow, it is convenient to rewrite the above relations as functions of a parameter, P, the shock strength defined by

$$P = \frac{p_1 - p_0}{p_0} \quad . \tag{1.3.4}$$

We then obtain from equations (1.3.2), the following relationships:

$$
(1.3.5)\quad
\begin{aligned}
&\text{a.}\quad \dot{\xi} - u_o = c_o\left(1+\frac{\gamma+1}{2\gamma}P\right)^{\frac{1}{2}} ,\\
&\text{b.}\quad u_1 - u_0 = \frac{c_o P}{\gamma}\left(1+\frac{\gamma+1}{2\gamma}P\right)^{-\frac{1}{2}} ,\\
&\text{c.}\quad c_1 = c_0\left[\frac{(1+P)\left(1+\frac{\gamma-1}{2\gamma}P\right)}{\left(1+\frac{\gamma+1}{2\gamma}P\right)}\right]^{\frac{1}{2}} ,\\
&\text{d.}\quad \rho_1 = \rho_0\,\frac{\left(1+\frac{\gamma+1}{2\gamma}P\right)}{\left(1+\frac{\gamma-1}{2\gamma}P\right)} .
\end{aligned}
$$

On substituting for p, ρ in terms of P, the equation of state for a polytropic fluid (1.1.4) yields the result for the change in entropy across the shock as

$$
\text{e.}\quad \exp\left(\frac{S_1-S_0}{c_v}\right) = (1+P)\left[\frac{\left(1+\frac{\gamma+1}{2\gamma}P\right)}{\left(1+\frac{\gamma-1}{2\gamma}P\right)}\right]^{-\gamma} .
$$

Since $S_1 \geq S_0$, it follows after some algebra, that P satisfies the inequality

$$
P \geq 0 ,
$$

that is, the pressure behind the shock is greater than that ahead of it if the shock is of non-zero strength.

We now derive some important properties of shock transitions.

(i) A shock of non-zero strength is compressive. This result follows simply from relation (d) for $P > 0$.

(ii) The change in entropy across a shock is of the third order in the shock strength. From relation (e) we have on differentiating w.r.t P,

$$\frac{d}{dP}\left(\frac{S_1 - S_0}{c_v}\right) = \frac{\gamma^2 - 1}{4\gamma^2} P^2 \left[(1+P)\left(1+\frac{\gamma+1}{2\gamma}P\right)\left(1+\frac{\gamma-1}{2\gamma}P\right)\right]^{-1} .$$

Since the term in the square bracket is of order unity, the result follows.

(iii) Disturbances which are propagated from behind the shock always overtake the shock and the shock itself always overtakes any disturbance ahead of it. We note from relation (a) that for $P \geq 0$,

(1.3.6) $$\dot{\xi} \geq u_o + c_o .$$

On adding (b) to (c) and then subtracting the sum from (a), we have,

$$\dot{\xi} - (u_1 + c_1) = c_o\left(1+\frac{\gamma+1}{2\gamma}P\right)^{-\frac{1}{2}}\left(1+\frac{\gamma-1}{2\gamma}P\right)^{\frac{1}{2}}\left[\left(1+\frac{\gamma-1}{2\gamma}P\right)^{\frac{1}{2}} - (1+P)^{\frac{1}{2}}\right]$$

$$\leq 0 ,$$

since $1 + \frac{\gamma-1}{2\gamma}P \leq 1 + P$, as $\gamma \geq 1$, by definition.

The velocity of the shock, $\dot{\xi}$, is therefore such that

a, $$u_0 + c_0 \leq \dot{\xi} \leq u_1 + c_1 \ .$$

As disturbances in the fluid are propagated with the local velocity of sound relative to the motion of the fluid, the above deduction follows.

(iv) The limiting form of a shock as its strength, P, tends to zero is a sound wave. As the shock strength tends to zero, we have from relations (b) and (c) of (1.3.5), the results

$$u_1 \to u_0 \text{ and } c_1 \to 0 \ .$$

Therefore, from relation (1.3.6 a), we obtain in the limit as $P \to 0$,

$$\dot{\xi} = u_0 + c_0 \ ,$$

that is, the flow speed relative to the shock front is the sound speed. This is the condition for the propagation of sound waves as given by the characteristics in the physical plane.

A problem which can be solved completely using the relations (1.3.5), is that of a piston being pushed into a fluid at rest with uniform velocity, u_1. A discontinuity in the flow immediately develops which is resolved as a shock front propagating into the stagnant fluid with velocity $\dot{\xi}$. Since u_1 is constant, relation (b) yields a quadratic equation for P of which the positive root is chosen. The solution is then completed by obtaining the remaining flow variables, all with constant values, from the appropriate relations of (1.3.5).

In the preceding example, the region behind the shock front is one of constant entropy. This is due to the motion of the piston being uniform in all respects. In general, however, the piston motion is usually a non-uniform function of the time and consequently the pressure change across the shock is non-constant and the region behind the shock wave is now one of variable entropy which considerably complicates the further development of the solution. It is with such flows that the approximate methods advocated later are concerned.

§4. THE HYDRAULIC ANALOGUE

An analogue to the motion of gases is encountered under certain circumstances in the motion of an incompressible fluid under the action of gravity and when there is a free surface. In particular, we are concerned with the motion of water in the problem analogous to the 'piston problem' of gas dynamics in a one-dimensional, unsteady system.

We assume that the water is contained in an infinitely long channel of constant depth in which there is a vertical plate free to move in a horizontal direction. For the basic principles, we follow Lamb [1952 Ed.] who assumes that the motion of the system may be determined according to the principles of "shallow water theory". This theory is derived from the normal hydrodynamic equation with the assumption that the excess of pressure over atmospheric pressure is as given by the hydrostatics law and it will be shown that the resulting equations of the system are analogous to those of the one dimensional, unsteady motion of a perfect polytropic gas in an isentropic state provided the adiabatic exponent is taken as 2.

Using this theory, Stoker [1948] has given an account of those hydrodynamical flows similar in type to the simple wave and shock wave flows of gas dynamics. In an appendix to this paper, Friedrichs has shown by a perturbation procedure that the hydrostatic law is valid so long as the height of the free surface above the bottom of the channel is small compared to the radius of curvature occurring at the free surface. This condition has subsequently been contested by Ursell [1953].

The following derivation of the governing equations is based on the assumption of the hydrostatics law and is similar to that of Stoker.

a. Basic Equations

An inviscid fluid of constant density, ρ, is contained in an infinitely long flat-bottomed channel and the motion of the fluid is assumed to be unsteady and in two dimensions. The choice of coordinate system is as indicated in Figure 6. The free surface of the fluid is given by the equation $y = h(x,t)$ and the components of velocity in the directions (x, y) are specified by (u, v) respectively. The y axis is chosen vertically upwards.

Under the assumption of constant density the general equations of continuity and motion, (1.1.6) and (1.1.7), reduce respectively to the simple forms,

a. Continuity: $u_x + v_y = 0$.

(1.4.1) b. Motion in the x-direction: $\rho \dfrac{Du}{Dt} = -p_x$.

c. Motion in the y-direction: $\rho \dfrac{Dv}{Dt} = -p_y - g\rho$,

where g denotes the acceleration due to gravity and p is the excess pressure above atmospheric pressure.

The boundary conditions imposed on the velocity are:

a. At the free surface, the vertical component, v, is given by

$$v = \frac{Dh(x,t)}{Dt} = h_t + uh_x, \quad \text{on } y = h(x,t) .$$

(1.4.2)

b. At the bottom of the channel, v is zero, that is

$$v = 0, \quad \text{on } y = 0 .$$

The dynamical condition on the pressure, p, at the free surface supplies the further boundary condition

(1.4.3) $$p = 0, \quad \text{on } y = h(x,t) .$$

If the equation of continuity is now integrated with respect to y for $0 \leq y \leq h(x,t)$, then on using the boundary condition, (1.4.2), on 0, we obtain the result,

$$h_t + uh_x + \int_0^{h(x,t)} u_x dy = 0 ,$$

which may be rewritten as

(1.4.4) $$h_t + \frac{\partial}{\partial x} \int_0^{h(x,t)} u\, dy = 0 .$$

At this stage, we introduce the approximation which leads to shallow water theory. We assume that the vertical component of acceleration of the fluid particles has no effect on the pressure, p. On integrating equation (1.4.1)b for p and using the boundary condition for the pressure at the free surface, it is seen that the above assumption is equivalent to assuming the hydrostatic law for the variation in pressure throughout a vertical 'column' of the fluid, that is

$$(1.4.5) \qquad p = \rho g(h-y) .$$

We note from this relation that

$$p_x = g\rho h_x ,$$

so that p_x is independent of y. It follows that the horizontal component of acceleration is the same for all fluid particles in a plane perpendicular to the direction of the x-axis. All particles which once lie in such a plane always do so, that is, the horizontal component of velocity, u, is a function of x and t only. In the subsequent work we shall assume that the fluid is initially in uniform motion in the x-direction, so that the above condition is satisfied.

The horizontal component of velocity, u, is then a function of (x, t) and the equation of motion, (1.4.1)b, may thus be written

$$(1.4.6) \quad \text{a.} \qquad u_t + uu_x = -gh_x ,$$

and the integrated form of the equation of continuity, (1.4.4), reduces to

$$\text{b.} \qquad h_t + uu_x + hu_x = 0 .$$

It is convenient to introduce the concepts of excess force per unit length and of mass per unit length, denoted respectively by $\bar{p}$ and $\bar{\rho}$, where

$$\bar{p} = \int_0^{h(x,t)} p \, dy$$

and

$$\bar{\rho} = \rho \int_0^{h(x,t)} dy \, .$$

On using the hydrostatic pressure law, these expressions integrate to yield

$$\bar{p} = \frac{g\rho h^2}{2}$$

and

$$\bar{\rho} = \rho h \, ,$$

that is, $\bar{p}$ and $\bar{\rho}$ satisfy the relation

$$\bar{p} = \frac{g}{2\rho} \bar{\rho}^2 \tag{1.4.7}$$

The relation between $\bar{p}$, $\bar{\rho}$ is thus similar to that between p, ρ for a polytropic gas in an isentropic state with adiabatic index, γ, taken as 2. There is, however, one important difference. In the latter system, the constant of proportionality in the equation of state is dependent upon the entropy and consequently the relation cannot be used across a shock front. In shallow water theory, however, this constant is 'universal' for the system so that the above restriction no longer holds.

The 'sound-speed', c, may be defined in the same manner as previously, that is

$$c^2 = \frac{d\bar{p}}{d\bar{\rho}} = \frac{g}{\rho}\,\bar{\rho} = gh \ , \tag{1.4.8}$$

and on substituting for $h(x,t)$ in terms of c, the equations of continuity and motion, (1.4.6), become

$$\begin{aligned} \frac{1}{2}u_t + \frac{1}{2}uu_x + cc_x &= 0 \ , \\ c_t + uc_x + \frac{1}{2}cu_x &= 0 \ . \end{aligned} \tag{1.4.9}$$

On comparing this system of equations together with the 'adiabatic law', (1.4.7), with those for a perfect polytropic gas with constant entropy, $S = S_o$, and adiabatic exponent 2, that is (1.1.8) with $\gamma = 2$, it is evident that they are formally equivalent. We may conclude that under similar conditions, flow patterns in both systems exhibit analogous features. Explicit results for simple waves in shallow water are therefore not given as they may be obtained from those derived in §2.

However, when entropy variations occur in the flow of a gas, as when shocks develop, this correspondence of results breaks down and the conditions appropriate to shallow water theory must be derived ab initio.

To distinguish the Riemann Invariants in the hydraulic analogue from their counterparts in gas dynamics, we define them as r and s where

$$(1.4.10) \qquad \begin{aligned} r &= \frac{u}{2} + c , \\ s &= -\frac{u}{2} + c . \end{aligned}$$

On a C^+ characteristic, given by $\frac{dx}{dt} = \frac{3}{2} r - \frac{1}{2} s$, we have the condition $r = \text{constant}$ and on a C^- characteristic, $\frac{dx}{dt} = \frac{1}{2} r - \frac{3}{2} s$, we have $s = \text{constant}$.

Equations (1.4.9) written in characteristic form are then

$$(1.4.11) \qquad \begin{aligned} & r_t + (\tfrac{3}{2} r - \tfrac{1}{2} s)\, r_x = 0 \\ \text{and} \quad & \\ & s_t + (\tfrac{1}{2} r - \tfrac{3}{2} s)\, s_x = 0 . \end{aligned}$$

b. <u>Bores</u>

We now derive the jump conditions appropriate to a surface of discontinuity in the hydraulic analogue. In hydrodynamics, as distinct from gas dynamics, it is customary to refer to these discontinuities as 'bores', and as the terminology is indicative of the fluid referred to, the convention is observed here.

The situation corresponding to the formation of a shock in shallow water theory, is effected by pushing a vertical plate horizontally through the water with non-decreasing velocity. The C^+ characteristics, along which the disturbances due to the motion of the plate are propagated, converge to form an envelope and the apparent many-valuedness of the flow is reconciled by a bore, which allows the dependent variables to change discontinuously. The jump conditions across the bore are obtained from the principles of conservation of mass and momentum in precisely the manner given in §3, and by comparison with the same are

$$\text{Conservation of Mass:} \quad \bar{\rho}_1(u_1 - \dot{\xi}) = \bar{\rho}_o(u_o - \dot{\xi}) = m \,,$$

(1.4.12)

$$\text{Conservation of Momentum:} \quad \bar{\rho}_1 u_1(u_1 - \dot{\xi}) - \bar{\rho}_o u_o(u_o - \dot{\xi}) = \bar{p}_o - \bar{p}_1 \,,$$

where the notation is as defined previously and 'm' specifies the mass flux through the bore.

An important difference in the mechanism of shocks and bores emerges if the principle of conservation of energy is applied to the flow. From (1.3.2) c with $\gamma = 2$, a measure of the change in energy of the fluid across the bore is provided by the expression

$$[\bar{\rho}_1 \{\tfrac{1}{2}u_1^2 + \frac{\bar{p}_1}{\bar{\rho}_1}\} (u_1 - \dot{\xi}) + \bar{p}_1 u_o] - [\bar{\rho}_o \{\tfrac{1}{2}u_o^2 + \frac{\bar{p}_o}{\bar{\rho}_o}\} (u_o - \dot{\xi}) + \bar{p}_o u_o] .$$

On substituting for m, u_o, $\bar{p}_1$ and $\bar{p}_o$ in terms of $\bar{\rho}_1$, $\bar{\rho}_o$ and m from relations (1.4.10) and (1.4.7), the above form can be written, after some manipulation, as

$$\frac{mg}{\rho} \frac{(\bar{\rho}_o - \bar{\rho}_1)^3}{4\bar{\rho}_1\bar{\rho}_o}$$

Since the quantities in this expression are all positive, the energy balance of the system is not maintained unless the motion is continuous, in which case $\bar{\rho}_1 = \bar{\rho}_o$ and the bore is then vanishingly weak. If we do not wish to postulate the existence of energy sources at the bore, so that we assume a particle loses energy on crossing a bore of finite strength, then in the above expression we require the condition

(1.4.13) $$\bar{\rho}_o \leq \bar{\rho}_1 ,$$

to hold in any possible transition.

Since $h(x, t)$ is a constant multiple of $\bar{\rho}$, the energy condition implies that particles move across the bore from a region of lower depth to one of higher depth, that is on adopting the convention used for shock waves, the bore front moves in the direction from the region 'behind' to the region 'ahead'. We assume that the region 'ahead' is the one in which x may increase. The velocity of the bore front $\dot{\xi}(t)$ is then positive.

The fact that the law of conservation of energy is violated at the bore is to be interpreted as meaning that the energy balance cannot be maintained through the sole action of mechanical forces and that the loss of mechanical energy is manifested as heat due to turbulence at the front of the bore. In compressible flows, the conversion of mechanical energy into heat is permitted through the agency of entropy and the conservation of energy is maintained.

A convenient parameter to choose to illustrate some properties of bore transitions is H, the ratio of the depths of water behind and ahead of the bore, that is

$$H = \frac{h_1}{h_o} = \frac{\bar{\rho}_1}{\bar{\rho}_o} .$$

From relation (1.4.13), we have the condition

$$H \geq 1 . \tag{1.4.14}$$

In shallow water theory it is convenient to use non-dimensional flow parameters. This is easily accomplished by referring the relevant equations to the constant value of the velocity of sound in the region ahead of the bore, which is a stagnation region in the problems which are discussed. The symbol 't' is then used to denote a measure of time having the dimensions of length. The stagnation velocity of sound is thus taken as unity.

The jump conditions in terms of H are then, from (1.4.8) and (1.4.12),

$$\text{a.} \qquad \dot{\xi} = H\sqrt{\frac{H(1+H)}{2}}$$

$$(1.4.15) \qquad \text{b.} \qquad u_1 = (H-1)\sqrt{\frac{1+H}{2H}}$$

$$\text{c.} \qquad c_1 = \sqrt{H}$$

The above equations are the analogues in shallow water theory to those which hold for shock waves. With methods similar to those employed previously we may deduce from the above equations that:

(i) the limiting form of a bore is a 'sound-wave',

(ii) disturbances propagated from behind the bore always eventually interact with the bore, and

(iii) the bore itself always overtakes any disturbances ahead of it.

CHAPTER II

STATEMENT OF THE PROBLEM AND FRIEDRICHS' APPROXIMATE SOLUTION

INTRODUCTION

We now state the general problem which the approximate methods presented in the subsequent chapters attempt to describe and give a summary of a well-known approximate solution due to Friedrichs [1948] which will be examined in the light of those methods.

In §3 Chapter I, it was indicated that the problem of a piston moving with uniform velocity along a tube of constant cross-section into a fluid at rest could be uniquely solved by an elementary application of the Rankine-Hugoniot relations. However, in the general case of piston motion, the velocity of the piston is non-constant and the resulting motion of the flow is difficult to determine. In particular, this thesis is concerned with the problems associated with the formation and decay of shocks in a polytropic gas and in water. The forward-facing characteristics from the piston path diverge (converge) thus generating a simple rarefaction (compression) wave which, by the results of §3 Chapter I, will eventually overtake the shock causing it to be modified. As the strength of the shock is now non-constant and the region ahead is uniform, the region in the (x, t)-plane bounded by the particle path through the initial point of interaction and the shock locus is one of variable entropy. Consequently the full equations, (1.1.8), must be used for a complete description of the motion. The flow-field is further complicated by the reflections at the piston path of the backward propagated pressure waves from the shock front which then give rise to a secondary interaction with the shock. It can be anticipated that a curved extension to the shock locus will develop, but neither its shape nor the flow pattern behind it can be expressed in terms of known functions. In such cases one usually resorts to numerical methods of integration, based, for example, on the use of the characteristics and the shock conditions. Another approach, the one taken here, is to obtain a solution for the flow-field by analytical methods which involve certain approximations which will be carefully noted as they arise.

As the particular problems with which we are concerned -- the formation and decay of shock waves -- have many aspects in common, we give in the following a detailed account of the nature of the flow-field for the case of the decay of a shock and then quickly describe the corresponding situation for the formation problem.

§1. DECAY OF A SHOCK

We assume that the piston is accelerated instantaneously from rest to a constant velocity u_1, thus producing a shock of constant strength p_1 which advances into the stagnation region with constant speed U. After a finite time the piston is retarded with its path in the (x, t)-plane given by

$$u_p = X_1(t_p) ,$$

where X_1 is a continuous and differentiable monotonic decreasing function satisfying the conditions

$$\text{(i)} \qquad X_1'(0) = u_1 ,$$

$$\text{(ii)} \qquad X_1''(t_p) \leq 0 , \quad t_p \geq 0 ,$$

which ensure that the transition of the piston motion from a uniform state to a retarded state is continuous and that the piston is decelerated.

The retardation of the piston generates a simple forward-facing rarefaction wave which eventually overtakes the constant strength shock and subsequently modifies it. The flow-field in the physical plane is as illustrated in Figure 7. We do not consider the secondary effects of the reflections of the backward propagated pressure waves off the piston.

In the regions ahead and behind the constant strength shock, parameters are specified by the suffices '0' and '1' respectively.

The flow-field consists in the early stages of the interaction of five distinct regions.

(i) The region bounded by the shock locus, SNA, and the line $t = t_a$, in which the fluid is at rest.

(ii) The triangular region bounded by the constant strength shock locus, NA, the locus of the uniform motion of the piston AO, and the final C^+ characteristic, ON, which leaves the piston during its uniform motion. The flow is one of constant properties: $u = u_1$, $c = c_1$, and $S = S_1$, and is obtained in terms of the constant velocity of the shock, U, and the local velocity of sound in the stagnation region ahead, c_o, by a simple application of the Rankine-Hugoniot shock relations.

(iii) The region bounded by the C^+ characteristic, ON, on which $\alpha = \alpha_1$, the C^- characteristic, NB, through the initial point of modification of the shock wave, and the portion of the piston path represented by the curve, OB. This region is adjacent to one of uniform conditions and as the piston velocity is decreasing, the flow in it is a forward-facing simple rarefaction wave for which the Riemann invariant, β, has a constant value, β_1. The flow variables may then be obtained in terms of the piston velocity, $X_1'(t_p)$ and the Riemann invariant β_1, in a manner similar to that given in §2 Chapter I.

Up to this stage, the flow behind the shock has been exactly represented either by a uniform region or by a simple wave. However, in the following regions the flow is now complicated by the effects of the interaction of the simple wave and the shock wave.

(iv) The region bounded by the particle path NE, and the final backward-facing C^- characteristic of the simple wave, NB. As the particle paths from the curved shock locus do not enter this region, the flow is isentropic with entropy $S = S_1$. However, this region is subject to the effects of the pressure waves reflected back from the shock along the C^- characteristics and consequently both α and β are variable. The motion is thus governed by the isentropic form of equations (1.1.8) and the boundary conditions are provided by the conditions first, that the flow variables on the bounding C^- characteristic, NB, are connected through the simple wave relations and second, that along the piston path the particle velocity u is a known function.

(v) The region bounded by the curved extension of the shock locus, NS, and the particle path, NE, through the initial point of interaction, N. Variations in entropy are propagated into this region along the particle paths from the shock locus. The flow is non-isentropic and the full system of equations (1.1.8) must be employed. The boundary conditions are provided by the conditions on the particle path NE and at the back of the shock locus NS. Along NE, the entropy is constant and is of the same value as in the region of uniform flow, that is $S = S_1$. The shock locus, NS, constitutes a 'floating' boundary of the region and consequently its representation in the physical plane, $x = \xi(t)$, is obtained as part of the solution for the flow in this region. Immediately behind it, the flow variables can be expressed as functions of the unknown shock velocity $\dot{\xi}(t)$ and the constant value of the speed of sound in the stagnation region ahead, c_o, from the Rankine-Hugoniot relations.

An important special case of this problem occurs when the piston after creating the shock of constant strength is suddenly stopped. The family of C^+ characteristics of the resulting simple wave forms a pencil of straight lines through the origin, taken for convenience at the point where the piston motion ceases. The flow-field is as illustrated in Figure 8 and is similar to that described above with the addition of a region, bounded by the tail of the simple wave and the 'piston path', which is specified by the suffix '2'. The flow in it is easily obtained. The region is crossed by the C^{+} characteristics of the simple wave for which the Riemann invariant β has the constant value β_1. Along the piston path as represented by the t-axis, the particle velocity is zero. On using the compatability condition on the C^{+} characteristics of the region in conjunction with the above conditions, the fluid velocity u_2 is found to be zero throughout the domain and the local speed of sound c_2 to be constant and given by $c_2 = (\gamma-1)\beta_1$. The C^{-} characteristic FB is therefore straight.

The simple wave region is then bounded by the C^{+} characteristics ON and OF on which the Riemann invariant α has the values α_1 and β_1 respectively.

It is important to note that when the whole history of the shock locus is considered the effects of the successive reflections at the piston path of the backward propagated pressure waves from the shock locus may lead to the formation of secondary shocks in the flow-field which will affect the behaviour of the modified shock NS. Due to the effects of the variations in entropy it cannot be shown conclusively that such shocks will form. If the constant strength shock is of moderate strength then we shall follow Lighthill [1950] and assume that should a secondary shock be formed it will only be after a very long time. Consequently, in the foregoing work the discussion of the decay of shock waves is limited to two cases. Firstly, when the constant strength shock wave is strong only the initial stages of the process of decay are examined and secondly, when the constant strength shock is weak then any secondary effects which may be introduced will be neglected so that the complete history of the shock locus may be described.

§2. FORMATION OF A SHOCK

The piston is now assumed to be pushed into the fluid at rest, with its path in the physical plane given by the relation

$$x_p = X(t_p) ,$$

where the function X satisfies the conditions of (1.2.10).

A forward-facing simple compression wave is generated, the C^+ characteristics of which are straight and, after a finite time, form an envelope indicating that the hypothesis of inviscid flow breaks down. As described in §3 Chapter I, a shock wave is formed at the initial point of the envelope, beginning with zero strength.

For the present we assume that the initial piston acceleration is non-zero so that the shock wave begins on the leading C^+ characteristic

$$x = c_o t$$

of the simple wave at the point $M(x_m, t_m)$ whose co-ordinates are given by (1.2.15), that is

$$x_m = \frac{2}{\gamma+1}\,\frac{c_o^2}{X''(0)} \quad ; \quad t_m = \frac{2}{\gamma+1}\,\frac{c_o}{X''(0)} \; .$$

The flow-field is then as represented in Figure 9 and consists of four distinct regions.

(i) The region of stagnation, SMOx, into which the shock wave advances.

(ii) The isentropic simple compression wave region, bounded by the initial C^+ characteristic OM, the portion of the piston path represented by OB and the C^- characteristic MB, through the initial point of formation of the shock. As this region is separated from the stagnation region ahead by the initial C^+ characteristic from the piston path, OM, the two regions have the same value of entropy, $S = S_o$.

(iii) The region of non-uniform isentropic flow, bounded by the particle path through the initial point of the shock, ME, the C^- characteristic MB, and the continuation of the piston path from the point β, in which the entropy has the same value as in the simple wave and stagnation regions. The boundary conditions for the governing equations, (1.1.8), with $S = S_o$, are given as before by the conditions on the piston path and on the bounding C^- characteristic of the simple wave.

(iv) The region of variable non-isentropic flow, bounded by the particle path ME, and the curvilinear shock locus MS, for which the full non-isentropic form of equations (1.1.8) must now be solved. As in the previous problem the shock locus MS is a floating boundary of the region. The boundary conditions are provided by the conditions that along the particle path NE, the entropy is given by its value in the stagnation region ahead of the shock and also that the flow variables immediately behind the shock locus satisfy the Rankine-Hugoniot equations.

When the initial acceleration of the piston is zero then, as was shown in §2 Chapter I, the envelope of the C^+ characteristics is formed in the simple wave domain and consequently the shock begins in this region, as is illustrated in Figure 10. Since disturbances which originate at the shock cannot be propagated directly into the region ahead, MCS, the simple wave solution may be extended across NC into this region. The point in the physical plane where the shock is first formed, represented by M, is found by solving equations (1.2.14) for the parameter λ, denoting the time when the C^+ characteristic through the point (x, t) started from the piston. This value of λ is then substituted in (1.2.13) to yield the required co-ordinates. Apart from this additional region, the flow-field is similar to that described above and consequently the details are not repeated.

The probable formation of secondary shocks can be anticipated, but as we are concerned in this problem only with the initial stages of the interaction their effects are outside the present investigation.

The general boundary value problem for the flow in the 'zone of penetration', BMS or BNS in Figures 9 and 7 respectively for the formation and decay of shocks may be formulated in the following manner.

A solution to the general system of equations (1.1.8) is sought in the region BMS , BNS which satisfies the jump conditions along the shock curve as yet undetermined, and the prescribed boundary conditions on the piston path and the final C^- characteristic of the incident simple wave.

The problem is of the type where floating boundary conditions are specified along the unknown shock locus, defined in the physical plane by the relationship

$$\frac{dx}{dt} = \dot{\xi}(t) ,$$

and no direct theoretical treatment seems possible, although the solution to each particular problem could be determined by numerical step by step methods as indicated previously. The most fruitful approach would seem to consist in constructing some approximate theory for the flow-field and in this way obtain a deeper insight into the respective roles of the dependent variables. This is the procedure adopted here.

The main purpose of the approximate methods which will be presented, is to determine the path of the shock in the physical plane. From physical considerations we would expect the shock curve to be determined uniquely by the above formulation and indeed we assume this although no proof of uniqueness is given.

§3. THE FORMATION AND DECAY OF BORES

The formation and decay of bores may be described in a manner similar to that given above in §§1, 2 for the corresponding problems in compressible flow with the simplification that now there is nothing to correspond with the role played by the variations in entropy. Consequently the diagrams of the flow-fields are similar to Figures 7 and 9 with the isentropic non-uniform region BNE , BME extended to cover the whole domain to the curved bore locus.

However, for the decay of a bore it can now be shown that secondary bores will develop unless the vertical plate, that is, the "piston" is withdrawn from the water with a speed sufficient to cause the incident simple wave to be completed by vacuum conditions. The flow behind the bore is then nowhere compressive.

We assume that the piston after producing the bore of constant strength is stopped impulsively at the origin. The flow-field is then as illustrated in Figure 11 in which the C^+ characteristics BB' , HH' represent respectively the reflections at the piston of the C^- characteristics NB , GH. The particle velocity on the piston is zero and consequently on this line $r = s$. As before, the flow in the region OFB is one of uniform conditions with $u_2 = 0$ and $c_2 = s_n$ where a suffix 'n' denotes that quantity is taken at the point 'N'. In particular, on the C^+ characteristic OFF' (the tail of the simple wave) we have $r = s_n$. Before proving that a secondary bore will form, we require to show that the Riemann invariant s is such that

(i) $$s_n \leq s \leq 1$$

(ii) $$\frac{ds}{dH} < 0 ,$$

where H is as defined in §4 Chapter I.

These relations are proved in §A Appendix I.

The proof that a secondary bore forms is in two stages:

(a) it is shown that the flow in the region F'FBB' is a backward-facing simple rarefaction wave, and

(b) it is then shown that the reflection of this wave at the piston path gives rise to a compression wave.

(a) The region F'FBB' is adjacent to one of constant properties, OFB, and is therefore a simple wave which is backward-facing as the Riemann invariant r has the constant value s_n. The C^- characteristics are then straight and determined by the equation

$$\left(\frac{dx}{dt}\right)_{C^-} = \frac{1}{2}s_n - \frac{3}{2}s .$$

With increasing time, the velocity of the bore $\dot{\xi}(t)$ decreases and thus from (1.4.15), H also decreases. From (ii) above, it then follows that s increases with increasing time and consequently $\left(\frac{dx}{dt}\right)_{C^-}$ then decreases with increasing time. Geometrical considerations then show that the gradient of the C^- characteristics decrease with increasing time and consequently they diverge from the C^+ characteristic FF' in the $x < o$ direction. The backward-facing simple wave is then a rarefaction wave.

(b) Let H' , J' be two points taken in the domain of the reflection of the simple rarefaction wave at the piston path. It is convenient to choose H' , J' such that they lie on a C^- characteristic, on which $s = s^x$, say. The situation is then as shown in Figure 11. To show that the wave reflected from the piston path is a compression wave, it is sufficient to prove that the C^+ characteristics HH' , JJ' converge. On the C^- characteristics JI , HG let s have the values s_1 , s_2

respectively, then from (a) it is evident that $s_2 > s_1$. At the points J, H on the piston path the values of r are s_1, s_2 respectively since $r = s$ on the piston path. Now, on the C^+ characteristic HH', $\frac{dx}{dt} = \frac{3}{2} r - \frac{1}{2} s = \frac{3}{2} s_2 - \frac{1}{2} s$ and consequently at H',

$$\left(\frac{dx}{dt}\right)_{HH'} = \frac{3}{2} s_2 - \frac{1}{2} s^x .$$

Similarly, at the point J' on the C^+ characteristic JJ',

$$\left(\frac{dx}{dt}\right)_{JJ'} = \frac{3}{2} s_1 - \frac{1}{2} s^x ,$$

$$\left(\frac{dx}{dt}\right)_{HH'} > \left(\frac{dx}{dt}\right)_{JJ'} , \quad \text{as } s_2 > s_1 .$$

Consequently the C^+ characteristics HH', JJ' will converge from the piston and after a finite time a secondary bore will form.

We therefore restate the problem of the decay of a bore by a point-centred simple wave such that the piston is now withdrawn from the water with a velocity $u = -2s_n$ or faster after producing the constant strength bore. The C^- characteristics reflected from the bore front will now tend to piston path asymptotically and no reflected wave will be formed.

The path of the piston is then given by

$$x_p = u_1 t_p \; ; \; u_1 > 0 \; ; \; t_p \leq 0$$

$$x_p = -2s_n t_p \; ; \; t_p > 0 \; ,$$

and the flow-field is as represented in Figure 12.

The incident simple wave is just completed by vacuum conditions and the tail of the wave on which $r = -s_n$ lies along the piston path OB. As the bore decays after a finite time to a 'sound-wave' in a stagnation region for which $r = 1 = s$, only the pencil of C^+ characteristics of the incident simple wave for which $r \geq 1$ will contribute to the process of decay.

In the problem of the formation of a bore, we are concerned, as before, only with the initial stages of the growth of the bore. Consequently, any secondary effects which occur may be neglected as being outwith the region of interest.

A detailed description of the flow-field for the problems in the hydraulic analogue is not given as it is closely associated with that given for the polytropic gas. The problems will be quickly described when an approximate solution to them is presented in Chapter III.

§4. FRIEDRICHS' APPROXIMATE THEORY

The simplest approximate solution to the problems formulated in the preceding section is provided by Friedrichs' 'simple wave' theory, [1948]. This theory is based upon the following facts.

(i) The change in entropy across a shock is of the order of the cube of the shock strength.

(ii) The change in the Riemann invariant β across a forward-facing shock is of order the cube of the shock strength. For a backward-facing shock, a similar property holds for the Riemann invariant α.

We shall develop the theory on the assumption that the shock is forward-facing, that is, the shock velocity $\dot{\xi}(t)$ is positive as the corresponding derivation for a backward-facing shock is similar.

The former of the above statements has been proved in §3 Chapter I, whilst the latter follows by formally expanding the shock relations (1.3.5) b and c, combined to give β as a function of the shock strength P, as a power series in P. The initial terms of the respective expansions can then be shown to be

$$\frac{s_1 - s_0}{c_o} = \frac{\gamma^2 - 1}{12\gamma^2} P^3 + O(P^4) ,$$

(2.4.1)

$$\frac{\beta_1 - \beta_0}{c_o} = \frac{(\gamma+1)^2}{64\gamma^3} P^3 + O(P^4) .$$

Thus if the strength of the shock is such that terms of the third order can be neglected, then the values of the Riemann invariant β and the entropy s ahead and behind the shock are the same. The shock wave is then by convention called 'weak'. If the above statements are applied to the problems of the decay of formation of shocks by simple waves then as the region behind the shock is one of constant entropy and Riemann invariant β, the flow in it is a forward-facing simple wave having straight C^+ characteristics on which the value of the Riemann invariant α is constant. Consequently this simple wave is merely the continuation of the incident simple wave and thus no secondary shocks form. To this order the shock wave does not alter the character of the simple wave. The value of the Riemann invariant α, found in terms of the shock speed $\dot{\xi}(t)$ and the quantities of the stagnation region ahead of the shock, at the 'back' of the shock locus is therefore equivalent to that of the simple wave taken at the shock locus. If we denote by suffices 'sw' and 'rh' quantities of the simple wave and shock wave respectively, then Friedrichs' assumptions for weak shocks imply the relation

(2.4.2) $$\alpha_{sw} = \alpha_{rh} ,$$

taken along the shock locus, defined by $dx = \dot{\xi}(t)\,dt$.

As the solution for the flow behind the shock is a simple wave, the above condition may be replaced by any other which identifies a flow quantity, or combination of such quantities, from the shock relations with a corresponding quantity of the simple wave, that is immediately behind the shock path we have

$$u_{sw} = u_{rh}\ , \quad c_{sw} = c_{rh}\ , \quad p_{sw} = p_{rh}\ . \tag{2.4.3}$$

It is important to note that on an exact theory for the interaction of a shock and simple wave, relations (2.3.2) and (2.3.3) are valid only at the initial point of interaction. Any of the above relations may be used to complete the solution for the flow-field by obtaining the equation of the shock path in the physical plane. To do this we require the expansion of the Riemann invariant α_{rh} in terms of the shock strength P to second order powers and also the corresponding expansion for the shock speed $\dot{\xi}(t)$. From the shock relations (1.3.5) a, b, c, the requisite expansions are seen to be

(2.4.4)

$$\text{(a)} \qquad \frac{\alpha_{rh} - \alpha_o}{c_o} = \frac{P}{\gamma}\left\{1 - \frac{\gamma+1}{4\gamma}P\right\} + O(P^3)\ .$$

$$\text{(b)} \qquad \frac{\dot{\xi}(t) - c_o}{c_o} = \frac{\gamma+1}{4\gamma}P\left\{1 - \frac{\gamma+1}{8\gamma}P\right\} + O(P^3)$$

As the adiabatic index of the fluid, γ, is greater than unity, the power series expansions (2.4.1) and (2.4.4) are valid for $P < 1$.

In the simple wave, the value of the Riemann invariant α_{sw} is given as a function of the space variables (x, t) by (1.2.7), that is

$$\{x - X(\alpha_{sw})\} = (\frac{\gamma+1}{2}\alpha_{sw} - \frac{3-\gamma}{2}\beta_i)\{t - T(\alpha_{sw})\} ,$$

where the functions X, T denote the parametric representation of the piston path and the suffix 'i' denotes that the quantity is measured at the point $I(x_i, t_i)$ which for the decay and formation problems is represented by the points $N(x_n, t_n)$ and $M(x_m, t_m)$ in Figures 7 and 9 respectively. Thus for the decay of the shock wave $\beta_i = \beta_1$ whilst for the formation of a shock wave $\beta_i = \beta_0$.

On using (2.4.2), the above relation when considered along the shock path, may be written in terms of α_{rh}, that is

$$(2.4.5) \quad \{x - X(\alpha_{rh})\} = (\frac{\gamma+1}{2}\alpha_{rh} - \frac{3-\gamma}{2}\beta_i)\{t - T(\alpha_{rh})\} .$$

It is convenient to refer the motion to a system of axes $(\overline{x, t})$ through the point I, that $\bar{x} = x - x_i$, $\bar{t} = t - t_i$. The path of the shock wave is then obtained by differentiating (2.4.5) with respect to $\bar{t}$ and using the condition $dx = \dot{\xi}(t)dt$. After some manipulation, the following ordinary differential equation is obtained for $\bar{t}$ as a function of α_{rh}.

(2.4.6)

(a) $\left[\frac{\gamma+1}{2}\alpha_{rh} - \frac{3-\gamma}{2}\beta_i - \dot{\xi}\right]\frac{d\bar{t}}{d\alpha_{rh}} + \frac{\gamma+1}{2}\bar{t} = -\frac{dF}{d\alpha_{rh}}$,

(b) where $F(\alpha_{rh}) = [X(\alpha_{rh}) - x_i - (\frac{\gamma+1}{2}\alpha_{rh} - \frac{3-\gamma}{2}\beta_i)\{T(\alpha_{rh}) - t_i\}]$.

This equation has to be solved subject to the initial condition,

$$\bar{t} = 0 \ , \ \alpha_{rh} = \alpha_{rh}(I) \ ,$$

where $\alpha_{rh}(I)$ is the value of α_{rh} at the initial point of modification of the shock.

To solve (2.4.6) we may either obtain the shock velocity $\dot{\xi}(t)$ as a function of α_{rh} from (1.3.5) or instead substitute for α_{rh} and $\dot{\xi}(t)$ in terms of the shock strength P from (2.4.4). We choose the latter method and after some algebra, obtain the following equation for $\bar{t}$ as a function of P.

(2.4.7) $$\frac{d\bar{t}}{dP} + \frac{2}{P}\{1 + \frac{\gamma+1}{8\gamma}P + O(P^2)\}t = -\frac{4\gamma}{\gamma+1.c_o.P}\{1+\frac{5(\gamma+1)}{8\gamma}P + O(P^2)\}\frac{dF}{dP} ,$$

with the initial condition $\bar{t} = 0$, $P = P_i$.

For the decay problem, P_i is non-zero; as the shock in the formation problem begins with zero strength, P_i is then zero.

Equation (2.4.7) is easily integrated and on fitting in the initial condition, the solution is

(2.4.8)

$$c_o(t-t_i) = -\frac{4\gamma}{\gamma+1}\,\frac{\{1-\frac{\gamma+1}{4\gamma}P+O(P^2)\}}{P^2}\int_{P_i}^{P} P\{1+\frac{7(\gamma+1)}{8\gamma}P^2+O(P^2)\}\,\frac{dF}{dP}\cdot dP \quad .$$

The solution corresponding to (2.4.8) for x is then obtained by substituting for t from (2.4.8) into (2.4.5) and the two expressions then provide the parametric representation of the shock path in terms of the shock strength P.

In deriving the solution for the shock locus, Friedrichs solves the differential equation corresponding to (2.4.7) exactly, that is the equation

$$\text{(2.4.9)} \qquad \frac{d\bar{t}}{dP} + \frac{2}{P(1-\frac{\gamma+1}{8\gamma}P)}\,\bar{t} = -\frac{4\gamma}{\gamma+1(c_oP)}\left[\frac{1+\frac{\gamma+1}{2\gamma}P}{1-\frac{\gamma+1}{8\gamma}P}\right]\frac{dF}{dP} \; .$$

As terms of the order P^3 are neglected by the hypotheses of the theory, this implies that in computing the difference

$[\frac{\gamma+1}{2}\alpha_{rh} - \frac{3-\gamma}{2}\beta_o - \dot{\xi}(t)]$ in (2.4.6), terms of order P^2 must be neglected when P is removed as a factor. Equation (2.4.9) ought then to be expanded to the significant terms in P which would then yield (2.4.7). The consequence of retaining the additional terms leads Friedrichs to give a result for the shock path which is different from that derived above. For the decay problem, Friedrichs then deduces an 'asymptotic' form for the shock locus by expanding the solution for small values of P. This 'asymptotic' form is identical to the result derived from (2.4.8) and indeed it would seem to be valid for all time and to be the significant solution for the shock path.

When the 'simple wave' theory is applied to the problem of the formation of a shock, then the solution as given by (2.4.8) is significant to terms of order P^2. This is easily proved. At the initial point of the envelope of the C^+ characteristics, where the shock begins, we have $\frac{dx}{d\alpha}$ and $\frac{dt}{d\alpha}$ both zero, which implies that $F'(\alpha_i)$ is also zero. If $F(\alpha)$ is expanded as a Taylor series around this point, then

$$F(\alpha) = \frac{(\alpha-\alpha_i)^2}{2} F''(\alpha_i) + O(\alpha-\alpha_i)^3 ,$$

since $$F(\alpha_i) = 0 .$$

Thus on substituting for $(\alpha-\alpha_i)$ in terms of P it follows that $F'(P)$ is of order P and therefore in (2.4.8), the solution for t is significant to terms of order P^2. However, for the decay problem, $F'(\alpha_i)$ is non-zero and consequently $F'(P)$ is of order unity and the shock path is then significant only to terms of order P.

Lighthill [1950] has examined the simple wave theory on the basis of an accuracy hypothesis and found that it predicts the correct behaviour of the flow-field to within the limits stated above. The approximate methods developed in the subsequent chapters neglect terms of a higher order in the shock strength than those neglected by Friedrichs, and will indicate a range of values of shock strength for which the 'simple wave' approximation is no longer applicable.

The application of the Friedrichs' approximation to the formation and decay of bores follows along lines similar to those given above although the algebraic details will differ. The solutions corresponding to particular piston motions will be evaluated in Chapter III.

CHAPTER III

§1. DERIVATION OF EQUATIONS AND BOUNDARY CONDITIONS IN THE CHARACTERISTIC PLANE

As there is nothing in shallow water theory which corresponds to the entropy in gas dynamics, the equations of the flow-field when written in characteristic form (1.4.11) yield a second order non-linear partial differential on elimination of either of the characteristic variables r, s. In the characteristic plane however, the system is described by a second order linear partial differential equation in either of the dependent physical variables x, t. In the variable t this equation is of the Euler-Poisson-Darboux type and has to be solved here in a finite domain of the (r, s)-plane which is bordered by the image of the back of the bore locus and by an s-characteristic. Consequently we expect the problems associated with the formation and decay of bores to be more tractable than the corresponding ones of gas dynamics.

The equations governing the system in the (r, s)-plane are obtained from the characteristic form of the equations in the physical plane (1.4.11) by interchanging the roles of the dependent and independent variables. After some algebra, the required equations are seen to be

$$x_s - \left(\frac{3}{2}r - \frac{1}{2}s\right)t_s = 0 \ ,$$
(3.1.1)
$$x_r - \left(\frac{1}{2}r - \frac{3}{2}s\right)t_r = 0 \ .$$

By eliminating either of the dependent variables x, t from (3.1.1), a second order linear partial differential equation is obtained. We eliminate the variable x and the resulting equation is

(3.1.2) $$(r+s)t_{rs} + \frac{3}{2}(t_r + t_s) = 0 \ .$$

Equation (3.1.2) is of the Euler-Poisson-Darboux type and every solution leads to a solution of (1.4.11) if the Jacobian of the transformation from the characteristic plane to the physical plane, $\frac{\partial(x,t)}{\partial(r,s)}$, is non-zero. By substituting for x_r, x_s in terms of t_r, t_s from (3.1.1) in the expression for the Jacobian it is seen that the correspondence of solutions breaks down whenever t_r, t_s is zero. We now develop the boundary conditions for the unique solution of (3.1.2) when applied to the problems of the formation and decay of bores.

The jump conditions across a hydraulic discontinuity may be written, from (1.4.15) with $u_o = 1$ and $c_o = 1$, in the forms

a. $$\dot{\xi} = \sqrt{\frac{H(1+H)}{2}} \ ,$$

(3.1.3) b. $$r = \sqrt{H} + \frac{(H-1)}{2}\sqrt{\frac{1+H}{2H}} \ ,$$

c. $$s = \sqrt{H} - \frac{(H-1)}{2}\sqrt{\frac{1+H}{2H}} \ .$$

Since $H \geq 1$ by definition, the above relations imply that $r \geq 1$ whilst s may be negative if H is large.

In a later part of this chapter we require the series expansions of (3.1.3) in terms of a parameter, which for convenience is chosen as σ, defined by

$$\sigma = \frac{2}{3}(u+c-1) = \frac{2}{3}\left(\frac{3}{2}r - \frac{1}{2}s - 1\right) . \tag{3.1.4}$$

With the above definition, the parameter σ could be used for the quantity $\frac{2}{3}\left(\frac{3}{2}r - \frac{1}{2}s - 1\right)$ at any point of the flow-field. However, we shall restrict its use to those points which constitute the locus of the bore. At such points the variables r, s are related by (3.1.3) so that it is possible to derive series expansions for r and s (and also $\dot{\xi}$) in terms of the parameter σ.

As the necessary derivation of these expansions is given in [6]B Appendix I, we merely quote here the requisite forms. For $\sigma < \frac{2}{3}$, the following series are convergent:

$$\text{a.} \quad r = 1 + \sigma - \frac{1}{64}\sigma^3 + \frac{1}{128}\sigma^4 - \frac{11}{12{,}288}\sigma^5 + O(\sigma^6) .$$

$$(3.1.5)\ \text{b.} \quad s = 1 - \frac{3}{64}\sigma^3 + \frac{3}{128}\sigma^4 - \frac{11}{4096}\sigma^5 + O(\sigma^6) .$$

$$\text{c.} \quad \dot{\xi} = 1 + \frac{3}{4}\sigma + \frac{5}{32}\sigma^2 - \frac{5}{128}\sigma^3 + \frac{3}{2048}\sigma^4 + \frac{51}{8192}\sigma^5 + O(\sigma^6) .$$

The path of the image of the bore, $P(r,s) = 0$, is determined by eliminating the parameter H from relations (b) and (c) of (3.1.3). After some manipulation, we obtain

$$(3.1.6) \qquad P(r,s) = 4\sqrt{2}\,(r^2 - s^2) - 4\}\sqrt{(r+s)^2+4} = 0 \ .$$

On this curve, we have

$$(3.1.7)\text{ a} \qquad \frac{ds}{dr} = -\frac{P_r}{P_s} = \frac{8\sqrt{2}\; r\sqrt{(r+s)^2+4} - (r+s)\{3(r+s)^2+4\}}{8\sqrt{2}\; s\sqrt{(r+s)^2+4} + (r+s)\{3(r+s)^2+4\}} \ ,$$

which on substituting for r, s in terms of H from (3.1.3) may be written in the form

$$\frac{ds}{dr} = -\left[\frac{H - \sqrt{\frac{1+H}{2}}}{H + \sqrt{\frac{1+H}{2}}}\right]^2 \ .$$

On the image of the bore locus, $\frac{ds}{dr}$ is then always negative except when H has the value unity corresponding to the limiting case of continuous flow. For this value of H we see from (3.1.3) that $r = 1 = s$ and from (3.1.6) it is evident that the point having these co-ordinates lies on the curve $P(r,s) = 0$. This point in the characteristic plane corresponds to the limiting form of the bore as the strength of the bore tends to zero, that is an 'acoustic wave'. In the decay problem the point (1.1) is then the 'point at infinity' whilst for the formation problem it represents the 'initial point'. At this point, we may also deduce from the above relation that the curve $P(r, s) = 0$ is tangential to the line $s = 1$.

On the bore locus we have the condition

$$\frac{dx}{dt} = \dot{\xi}(t) \quad ,$$

which supplies the boundary condition on $P(r, s) = 0$ for the Euler-Poisson-Darboux equation, (3.1.2).

Using relations (3.1.1), the above condition may be rewritten in the characteristic plane as

$$(3.1.7)\,b \qquad \frac{dr}{ds} = \left[\frac{\frac{3}{2}r - \frac{1}{2}s - \dot{\xi}(r,s)}{\dot{\xi}(r,s) - \frac{1}{2}r + \frac{3}{2}s}\right] \left(\frac{t_s}{t_r}\right) \quad .$$

The velocity of the bore $\dot{\xi}$ as a function of r, s is determined from (3.1.3) and is given by

$$(3.1.8) \qquad \dot{\xi}(r,s) = \frac{(r+s)}{4\sqrt{2}} \sqrt{(r+s)^2 + 4} \quad .$$

By substituting for $\frac{dr}{ds}$ and $\dot{\xi}(r, s)$ from (3.1.7) and (3.1.8) respectively, the above boundary condition for (3.1.2) on $P(r, s) = 0$, is seen to be

$$(3.1.9) \qquad \frac{t_s}{t_r} = G(r,s) \quad ,$$

where

(3.1.10)

$$G(r, s) = \left[\frac{\frac{1}{2}r + \frac{3}{2}s + \frac{(r+s)}{4\sqrt{2}}\sqrt{(r+s)^2+4}}{\frac{3}{2}r - \frac{1}{2}s - \frac{(r+s)}{4\sqrt{2}}\sqrt{(r+s)^2+4}}\right] \left[\frac{(r+s)\{3(r+s)^2+4\} + 8\ \ 2\ s\sqrt{(r+s)^2+4}}{(r+s)\{3(r+s)^2+4\} - 8\ \ 2\ r\sqrt{(r+s)^2+4}}\right]$$

The remaining boundary condition for (3.1.2) is provided by the condition that along the bounding C^- characteristic of the incident simple wave, the time t is a known function of the Riemann invariant r. We now derive this condition for the two types of flow, the formation of a bore and the decay of a bore.

We shall assume that in the former flow, the piston is pushed into the still water with a constant non-zero acceleration $a > 0$ and that the piston starts from rest. After a time $t = t^x$, the piston continues with uniform velocity $u^x = a\,t^x$, t^x being chosen such that only one envelope of the C^+ characteristics is formed by the motion of the piston. The flow-field is then as illustrated in Figure 13. Since the acceleration of the piston in the interval $o \leq t \leq t^x$ is constant, the bore is formed on the leading C^+ characteristic of the simple compression wave, $x = t$, beginning at the point M with co-ordinates, determined from (1.2.15),

(3.1.11) a $$x_m = \frac{2}{3a} = t_m\,.$$

The path of the piston is given by

$$x_p = \frac{1}{2} a t_p^2 \quad , \quad t_p \le t_p^x \quad ,$$

$$x_p = x_p^x + a t_p^x [t_p - t_p^x] \quad , \quad t_p > t_p^x \quad ,$$

and the Riemann invariant s of the simple wave has the constant value unity. x_p and t_p when regarded as functions of r are given by

$$(3.1.11)\text{b} \qquad x_p = \frac{(r-1)^2}{2a} \quad , \quad t_p = \frac{(r-1)}{a} \quad .$$

From (1.2.8)b with $\gamma = 2$, we then obtain the differential equation for the C^- characteristics of the simple wave which for the particular C^- characteristic through M is solved with the initial condition

$$r = 1 \text{ when } t = t_m .$$

On performing this integration, the required solution for time on the bounding C^- characteristic MB as a function of r, is

$$(3.1.12) \qquad t = \frac{t_m}{5} [16\sqrt{2}\ (1+r)^{-\frac{3}{2}} + 3(2r-3)] \quad .$$

For the decay problem, the piston motion is discontinuous at the origin, in the case considered here, and is given by

$$x_p = u_1 t_p \quad ; \quad u_1 > 0 \quad , \quad t_p < 0$$

$$x_p = -2 s_n t_p \qquad\qquad t_p > 0 \quad ,$$

so that the resulting flow due to the simple rarefaction wave is non-compressive. The statement of the decay problem is as given in §3 Chapter II and the condition on the bounding C^- characteristic of the incident simple wave domain, corresponding to (3.1.12), is easily found to be

$$t = t_n \left(\frac{r + s_n}{r_n + s_n}\right)^{-\frac{3}{2}} . \tag{3.1.13}$$

Figures 14, 15 illustrate respectively the images of the flow-fields in the characteristic plane for the formation and decay of a bore. The boundary-value problem for the decay of a bore is then as follows.

The time $t(r, s)$ is governed by (3.1.2) and a solution is sought in the finite region of the characteristic plane bordered by the images of the bore locus $P(r, s) = 0$, the simple rarefaction wave $s = s_n$ and the final C^+ characteristic which contributes to the process of decay of the bore $r = 1$. The boundary conditions are provided by the condition (3.1.9) on the curve $P(r, s) = 0$ and by (3.1.13) on the line $s = s_n$. The domain in the physical plane corresponding to the above region is illustrated in Figure 12 and is represented by MY_1, NM. The motion of the piston has been so arranged that no secondary bores will form, the proof being given in Chapter II. Consequently the region of validity of the solution will be as described above, that is the solution is valid for all t in the domain NY_1, M.

The corresponding boundary-value problem for the formation of a bore may be formulated in a manner similar to that given above with the modification that now the image of the incident simple wave is given by the line $s = 1$ and the corresponding boundary condition is given by (3.1.12). In solving the two boundary value problems the series expansions of the bore relations will be used and consequently the maximum value of σ on the bore locus must be such that $\sigma < \frac{2}{3}$. For the decay problem, this restriction has the effect of limiting the solution to bores which satisfy this condition. For the formation problem, the solution for time on the bore locus will be valid only to that portion of the bore locus for which $\sigma < \frac{2}{3}$ and consequently the general solution for $t(r, s)$ must be limited to the region bounded by the initial C^- characteristic of the simple wave, the portion of the bore locus for which $\sigma < \frac{2}{3}$ and the C^+ characteristic which intersects the bore locus at the point where the maximum value of σ occurs.

The general boundary value problem is not a classical one in which two data are given on a non-characteristic curve, but a variation with one datum, (3.1.9), given on this curve, and one on a characteristic bounding the region in which the solution is sought, that is either (3.1.12) or (3.1.13) according to the particular problem.

§2. FRIEDRICHS' SOLUTION

We now apply Friedrichs' 'simple wave' theory to the problems formulated above. In either problem, the flow behind the bore locus is the continuation of the incident simple wave and consequently no secondary bores will develop. The regions of validity of the respective solutions are as described above. For the formation and decay of bores we can derive, in a manner similar to that given for the corresponding problems associated with shocks in Chapter II, the differential equation which holds on the bore path, corresponding to (2.4.6), for $(t-t_i)$ as a function of the Riemann invariant r. The interpretation for t_i is as given in Chapter II. From equation (2.4.6), with $\gamma = 2$ and $s_o = 1$, we may write the following equation which is true on the bore locus.

$$[\frac{3}{2}r - \frac{1}{2} - \dot{\xi}(r)] \frac{d(t-t_i)}{dr} + \frac{3}{2}(t-t_i) = -\frac{dF}{dr} ,$$

$$\text{(3.2.1)} \qquad \text{where} \quad F(r) = [X(r) - x_i - (\frac{3}{2}r - \frac{1}{2})\{T(r) - t_i\}] .$$

It is convenient to rewrite this equation in terms of the parameter σ, defined by (3.1.4). On using the requisite expansions of (3.1.5), we then obtain

$$\text{(3.2.2)} \quad \frac{d(t-t_i)}{d\sigma} + \frac{2}{\sigma}[1 + \frac{5}{24}\sigma + 0(\sigma^2)](t-t_i) = -\frac{4}{3}\frac{[1 + \frac{5}{24}\sigma + 0(\sigma^2)]}{\sigma}\frac{dF}{dr} .$$

For the decay of a bore $t_i \equiv t_n$ and also as the incident simple wave is point-centred at the origin, the functions X and T are both zero. Consequently, we have

$$(3.2.3)\text{ a.} \qquad \frac{dF}{d\sigma} = -\frac{3}{2}t_n \ .$$

The initial condition for time t on the bore locus in this problem for (3.2.2) is

$$\text{b.} \qquad \sigma = \sigma_n \text{ at } t = t_n \ .$$

However, for the formation of a bore, $t_i \equiv t_m = \frac{2}{3a}$ on using (3.1.11)a. The functions X and T in this case are given by (3.1.11)b. After some algebra, it is found that

$$(3.2.4)\text{ a.} \qquad \frac{dF}{dr} = -\frac{2\sigma}{a} + 0(\sigma^3) \ .$$

The initial condition for (3.2.2) is now $\sigma = 0$ at $t = t_m$.

On integrating equation (3.2.2) for each type of flow and applying the relevant initial conditions, the required solutions, after some manipulation, can be written in the forms:

$$(3.2.5)\text{ a. Decay of a bore.} \quad \frac{t}{t_n} = \left(\frac{\sigma_n}{\sigma}\right)^2 \left[1 + \frac{5}{12}(\sigma_n - \sigma) + 0(\sigma_n{}^2)\right] \quad ,$$

$$\text{b. Formation of a bore.} \quad \frac{t}{t_m} = \left[1 + \frac{4}{3}\sigma + \frac{5}{72}\sigma^2 + 0(\sigma^3)\right]$$

The corresponding solutions for x as a function of σ may now be obtained by substituting for t from the above expressions in terms of σ in the relevant forms of (2.4.5).

§3. GENERAL SUMMARY OF PROCEDURE

The boundary value problems formulated in §1 of this chapter are solved and series representations for x and t are obtained in terms of the parameter σ. The method of solution consists essentially of two stages:

(a) The reduction of the Euler-Poisson-Darboux with the appropriate boundary conditions to an integral equation for time $t(r, s)$ on the bore path. On this curve r and s are related by means of the equation $P(r, s) = 0$. The integral equation is of the second order, non-homogeneous Volterra type (Tricomi 1957). To accomplish this, the reduction may be effected either by Riemann's method or by a variation of that method due to Martin (1943). The former method is chosen for two reasons. Firstly, Martin's variation introduces a function, the resolvent, corresponding to the Riemann function. The resolvent involves Appell's hypergeometric function of two variables and is more cumbersome in this instance than the normal hypergeometric function of Riemann's procedure. Secondly, Martin's method leads to an integral equation for either of the partial derivatives t_s or t_r on the bore locus.

Although knowledge of these quantities will be sufficient to determine t on the bore locus, Riemann's procedure is preferable as we may derive immediately an integral equation for $t(r, s)$ on the bore locus with no intermediate stage.

(b) Owing to the complicated nature of the kernals of the integral equations for the formation and decay problems, an exact solution is unobtainable and recourse is made to an approximate method to determine a suitable series representation for t on the bore locus as a function of σ. We could, of course, determine numerical solutions of the integral equations by well-known methods (Kunz 1957) but as we are primarily interested in deriving an analytic solution for the respective flow-fields this approach is not developed. In the formation problem, the procedure adopted is to assume a series expansion for t on the bore locus in terms of σ and the unknown coefficients of σ are then determined by equating similar powers of σ in the integral equation. By this method $t(\sigma)$ has been found to terms of order 4 in σ. In principle, higher order terms in this expansion could be determined but with a corresponding increase in the amount of algebraic detail. As noted before, the use of the series expansions in σ of the bore relations is permissable only to the portion of the bore where the maximum value of $\sigma < \frac{2}{3}$, and it is reasonable to expect the series expansion for $t(\sigma)$ to be convergent for such values of σ. For the decay problem two methods of solution are given. The first is obtained by deriving an integral equation in a manner similar to that employed in the formation problem. The main difference in the integral equations for the two flow-fields is that the latter is singular when $r = 1$. This behaviour is to be expected as we desire to determine the complete history of the bore path in the physical plane. As before, the kernel of the integral equation is complicated and recourse is made to an approximate procedure which consists of substituting for the dependent variables in terms of σ and neglecting terms of order 4. The reason for this is to minimise the amount of algebraic detail whilst still illustrating the general principle of the method. In this manner the integral equation can be transformed to an equivalent ordinary differential equation which is easily solved to yield the required solution for t on the bore path as a function of σ and σ_n, the value of σ at the initial point of modification of the bore. As terms of order σ_n^4 are neglected, the solution is valid only for bores which are not too strong. Numerical results are then given for the formation and decay problems emphasize the fact that the 'simple wave' theory of Friedrichs overestimates the rate at which the intensity of the bore increases in the formation problem and decreases in the decay problem. By comparison with the higher order approximation it is then possible to give an indication of the range of application of the 'simple wave' theory. Finally, a method based on the focussing equations of Meyer (1956) is used to determine the initial stages of decay of a bore of any strength. This method has the double advantage over that of the integral equation as the solution for any 'fixed' segment of the bore locus can be determined to any required degree of accuracy with very little difficulty and also that the solution for the flow in the region behind the bore locus is given in a more tractable form than the corresponding solution from the integral equation.

§4. COMPLETE 4TH ORDER SOLUTION FOR THE FORMATION OF A BORE

We are to solve the equation

(3.4.1) $$(r+s)t_{rs} + \frac{3}{2}(t_r + t_s) = 0 ,$$

with the boundary conditions:

(3.4.2) (i) On $s = 1$, $$t = \frac{2}{15a}\left[16\sqrt{2}\,(1+r)^{-\frac{3}{2}} + 3(2r-3)\right] ,$$

(ii) On $P(r, s) = 0$, $t_s = G(r, s)\, t_r$,

in the domain of the (r, s)-plane representing the image of the flow-field and shown in Figure 14.

Throughout this chapter, the following convention is observed. A Riemann function is a function of four variables, the 'current' co-ordinates r, s and the 'field' co-ordinates r_o, s_o. In the present application, the field point Y in Figure 14 will always lie on the line $s = 1$ and we take its co-ordinates to be $(r_\theta, 1)$. Thus r_θ is the r-co-ordinate of the point X on the path of the bore and we use s_θ to denote the corresponding s-co-ordinate so that $P(r_\theta, s_\theta) = 0$.

Equation (3.4.1) is a member of a class of equations which have been studied in detail by Darboux (1888) and for which the Riemann functions are known. In particular for (3.4.1) the Riemann function is given by

$$(3.4.3) \qquad W(r, s, r_\theta, 1) = \frac{(r+s)^{\frac{3}{2}}}{(1+r_\theta)^{\frac{3}{2}}} F(-\tfrac{1}{2}, \tfrac{3}{2}, 1, z) \; ,$$

$$\text{where} \qquad z = -\frac{(r-r_\theta).(s-1)}{(r+s).(1+r_\theta)} \; ,$$

and the standard notation for the hypergeometric function is used. By a simple application of Riemann's method we can now write down

(3.4.4)

$$t(Y) = \frac{1}{2}[\{Wt\}(M) - \{Wt\}(X)]$$
$$+ \int_M^X [\{\frac{3Wt}{2(r+s)} + \frac{1}{2}(Wt_r - tW_r)\} dr - \{\frac{3Wt}{2(r+s)} + \frac{1}{2}(Wt_s - tW_s)\} ds] \; ,$$

where the curvilinear integral is taken along the image of the bore locus, $P(r_\theta, s_\theta) = 0$.

In (3. 4. 4) none of the quantities t, t_r or t_s is known on $P(r, s) = 0$. We have merely relation (3. 4. 2) connecting t_r and t_s. However, from (3. 4. 2) $t(Y)$ is known and consequently (3. 4. 4) can be regarded as an integral equation for time t at the point X on the image of the bore locus. Before deriving this integral equation, it is found convenient to employ instead of W a function w which is $(r+s)^{-1}$ times the classical Riemann function W, and which can be written as

$$w = \frac{(r+1)^{\frac{1}{2}} (r_\theta + s)^{\frac{1}{2}}}{(1+r_\theta)^2} F(-\tfrac{1}{2}, -\tfrac{1}{2}, 1, p) \; , \tag{3.4.5}$$

$$\text{where} \quad p = \frac{(r - r_\theta).(s-1)}{(1+r).(r_\theta + s)} \; .$$

That w may be written in the above form is shown in §D Appendix I. This change of function is not absolutely necessary for the further development of the solution. However, if the function w as defined above is used then the following analysis is closely allied to the work of Pillow (1949) who whilst developing an approximate theory for the formation of shocks used a Riemann function of which the function w is a special case. This theory will be discussed in Chapter V.

In terms of w, (3. 4. 4) after some rearrangement of terms, becomes

(3.4.6)

$$t(Y) = \frac{1}{2}[\{(r+s)wt\}(M) + \{(r+s)wt\}(X)]$$

$$+ \int_M^X [t\{w(1-\frac{ds}{dr}) - \frac{r+s}{2}(w_r - \frac{ds}{dr}w_s)\} + \frac{(r+s)}{2}w(tr - \frac{ds}{dr}t_s)]\, dr .$$

On the curve $P(r,s) = 0$, t_r and t_s are related by (3.4.2) and consequently the final term in the curvilinear integral can be replaced by a form in which the only derivative to appear is the total derivative $\frac{dt}{dr}$ on the curve $P(r,s) = 0$, that is, after some algebra, we may write

$$t_r - \frac{ds}{dr}t_s = 2[\frac{r-s-\dot{\xi}}{r+s}]\frac{dt}{dr} .$$

By substituting the above expression into (3.4.6) and then integrating by parts to eliminate $\frac{dt}{dr}$, we obtain

(3.4.7)

$$t(Y) = [\frac{3}{2}r - \frac{1}{2}s - \dot{\xi})wt](X) - [(\frac{1}{2}r - \frac{3}{2}s - \dot{\xi})wt](M)$$

$$+ \int_1^{r_0} t[w(1-\frac{ds}{dr}) - \frac{d}{dr}\{(r-s-\dot{\xi})w\} - \frac{r+s}{2}(w_r - \frac{ds}{dr}w_s)]\, dr .$$

In the above equation, $t(Y)$ is known from (3.4.2) and w is completely determined from (3.4.5). In fact, the explicit representation of the various quantities of (3.4.7), with the exception of the integral, are:

(3.4.8)

$$t(Y) = \frac{2}{15a}[16\sqrt{2}\ (1+r_\theta)^{-\frac{3}{2}} + 3(2r_\theta - 3)] \ .$$

$$[(\tfrac{3}{2}r-\tfrac{1}{2}s-\dot{\xi})wt](X) = [\{\tfrac{3}{2}r_\theta - \tfrac{1}{2}s_\theta - (r_\theta - s_\theta)\}(1+r_\theta)^{-\frac{3}{2}}(r_\theta+s_\theta)^{\frac{1}{2}}]\, t(r_\theta, s_\theta) \ .$$

$$[(\tfrac{1}{2}r-\tfrac{3}{2}s-\dot{\xi})wt](M) = -\frac{2}{3a}[2\sqrt{2}\ (1+r_\theta)^{-\frac{3}{2}}] \ .$$

When the substitutions from (3.4.8) are made in (3.4.7) and the terms rearranged, the following integral equation for time $t(r_\theta, s_\theta)$ on the bore locus is obtained.

(3.4.9)

$$[\{\tfrac{3}{2}r_\theta - \tfrac{1}{2}s_\theta - \dot{\xi}(r_\theta, s_\theta)\}(r_\theta+s_\theta)^{\frac{1}{2}}(1+r_\theta)^{\frac{3}{2}}]\, t(r_\theta, s_\theta)$$

$$= \frac{2}{5a}[2\sqrt{2}\ (1+r_\theta)^{-\frac{3}{2}} + (2r_\theta - 3)]$$

$$- \int_1^{r_\theta} t(r,s)\,[w\frac{d\dot{\xi}(r,s)}{dr} - \{(\tfrac{3}{2}r-\tfrac{1}{2}s-\dot{\xi})w_r + (\tfrac{1}{2}r-\tfrac{3}{2}s-\dot{\xi})w_s\frac{ds}{dr}\}]\, dr \ .$$

The above equation is a non-homogeneous Volterra integral equation of the second type (Tricomi 1957). Inspection of the kernel shows that it is a bounded function in the range $1 \leq r \leq r_\theta$. Everything (except t) outside the integral is known as a function of r_θ while the square bracket inside the integral is a known function of r through the relation $P(r, s) = 0$. It should be noted that (3.4.9) provides an exact relation for $t(r_\theta, s_\theta)$ as no approximations have been made thus far. Consequently, the general solution if determinate will be valid for any stage in the process of formation of the bore. Unfortunately, it does not seem possible to obtain such a solution owing to the complicated nature of the integrand. However, as r_θ and s_θ are known in terms of the parameter σ to any required degree of accuracy from (3.1.5), an approximate solution may be obtained by assuming a power series expansion for t in this variable and then equating similar powers of σ in (3.4.9) after the integrand has been expanded. We have, in fact, found t to the fourth power in σ by this means.

The series solution for (3.4.9) is now developed. It is convenient to use the symbol ϵ to signify the value of σ at the point X on $P(r, s) = 0$. Relation (3.4.9) then becomes

(3.4.10)

$$[\{\tfrac{3}{2} r_\theta - \tfrac{1}{2} s_\theta - \dot{\xi}(r_\theta, s_\theta)\}(r_\theta + s_\theta)^{\frac{1}{2}}(1+r_\theta)^{-\frac{3}{2}}]\, t(\epsilon)$$
$$= \frac{2}{5a}[2\sqrt{2}\,(1+r_\theta)^{+\frac{3}{2}} + (2\sqrt{2}\,(1+r_\theta)^{-\frac{3}{2}} + (2r_\theta - 3)]$$

$$-\int_{\theta}^{\epsilon} t(\sigma)[w\frac{d\dot{\xi}(r,s)}{dr} - \{(\frac{3}{2}r-\frac{1}{2}s-\dot{\xi})w_r + (\frac{1}{2}r-\frac{3}{2}s-\dot{\xi})w_s\frac{ds}{dr}\}]\frac{dr}{d\sigma}\,d\sigma \quad .$$

As there is a considerable amount of algebraic detail involved in obtaining the necessary expansions of the various terms of (3.4.10), only the final result for each term as a function of σ and ϵ is quoted. The derivations of the respective expansions are, however, given in §C Appendix I.

The time $t(r,s)$ on the image of the bore locus is assumed to be of the form

$$(3.4.11) \qquad t(r,s) = \frac{2}{3a}[1 + a_1\sigma + a_2\sigma^2 + a_3\sigma^3 + a_4\sigma^4 + 0(\sigma^5)] \ ,$$

in which the coefficients a_i, $i = 1, 2, 3, 4$, are to be determined. When the strength of the bore tends to zero, σ also tends to zero and from the above expansion it is seen that $t(r,s)$ tends to $\frac{2}{3a}$, the initial point of the envelope formed by the C^+ characteristics of the incident simple wave at which we assume the bore to begin.

On substituting for r, s and $\dot{\xi}(r,s)$ in terms of σ from the relations (3.1.5), we obtain after some manipulation the following expansions.

(3.4.12)

a. $$\frac{ds}{dr} = -\frac{9}{64}\sigma^2\left[1 - \frac{2}{3}\sigma + \frac{49}{288}\sigma^2\right] + O(\sigma^5) \quad ,$$

b. $$\frac{d\dot{\xi}(r,s)}{dr} = \frac{3}{4}\left[1 + \frac{5}{12}\sigma - \frac{7}{64}\sigma^2 - \frac{1}{256}\sigma^3 + \frac{57}{2048}\sigma^4\right] + O(\sigma^5) \quad ,$$

c. $$\left[\left(\frac{3}{2}r - \frac{1}{2}s - \dot{\xi}\right)w_r + \left(\frac{1}{2}r - \frac{3}{2}s - \dot{\xi}\right)w_s\frac{ds}{dr}\right]$$

$$= \frac{3}{16}w\sigma\left[1 - \frac{\sigma}{3} + \frac{19\sigma^2 - 18\sigma\epsilon}{64}\right.$$

$$\left. - \frac{296\sigma^3 - 234\sigma^2\epsilon - 216\epsilon^2\sigma}{1536} + O(\sigma^4)\right] \quad ,$$

d. $$\frac{dr}{d\sigma} = \left[1 - \frac{3}{64}\sigma^2 + \frac{1}{32}\sigma^3 - \frac{55}{12,288}\sigma^4\right] + O(\sigma^5) \quad .$$

e. $$w(r, s, r_\theta, 1) = \frac{1}{2}\left[1 + \frac{\sigma - 3\epsilon}{4} \quad \frac{15\epsilon^2 - 6\epsilon\rho - \sigma^2}{32} \quad \frac{67\epsilon^3 - 30\epsilon^2\sigma - 6\epsilon\sigma^2 + 2\sigma^3}{256}\right.$$

$$\left. + \frac{546\epsilon^4 - 286\epsilon^3\sigma - 63\epsilon^2\sigma^2 + 60\epsilon\sigma^3 - 5\sigma^4}{4096}\right] + O(\sigma^5) \quad .$$

It then follows from relations (b), (c) and (d) above that

f. $$\left[w\frac{d\dot{\xi}(r,s)}{dr} - \left\{\left(\frac{3}{2}r - \frac{1}{2}s - \dot{\xi}\right)w_r + \left(\frac{1}{2}r - \frac{3}{2}s - \dot{\xi}\right)w_s\frac{ds}{dr}\right\}\right]\frac{dr}{d\sigma}$$

$$= \frac{3}{8}\left[1 + \frac{5\sigma - 9\epsilon}{12} + \frac{15\epsilon^2 + 10\epsilon\sigma - 2\sigma^2}{32} + \frac{30\epsilon\sigma^2 + 50\epsilon^2\sigma - 22\sigma^3 - 67\epsilon^3}{256}\right.$$

$$\left. + \frac{1638\epsilon^4 - 690\epsilon^3\sigma - 675\epsilon^2\sigma^2 - 438\epsilon\sigma^3 + 787\sigma^4}{12284}\right] + O(\sigma^5) \quad .$$

We also have from the expansions given by (3.1.5), the representations

(3.4.13)

a. $$[\{\frac{3}{2}r_\theta - \frac{1}{2}s_\theta - \dot{\xi}(r_\theta, s_\theta)\}\ (r_\theta + s_\theta)^{\frac{1}{2}}\ (1+r_\theta)^{-\frac{3}{2}}]\ t(\theta)$$

$$= \frac{\epsilon}{4a}\ [1 + (a_1 - \frac{17}{24})\epsilon + (a_2 - \frac{17}{24}a_1 + \frac{13}{32})\epsilon^2$$

$$+ (a_3 - \frac{17}{24}a_2 + \frac{13}{32}a_1 - \frac{107}{512})\epsilon^3 + (a_4 - \frac{17}{24}a_3 + \frac{13}{32}a_2 - \frac{107}{512}a_1 + \frac{145}{3072})\epsilon^4] + O(\epsilon^6)\ .$$

b. $$\frac{2}{5a}[2\sqrt{2}\ (1+r_\theta)^{-\frac{3}{2}} + (2r_\theta - 3)] = \frac{\epsilon}{2a}\ [1 + \frac{3}{8}\epsilon - \frac{15}{64}\epsilon^2 + \frac{61}{512}\epsilon^3 - \frac{473}{10240}\epsilon^4] + O(\epsilon^6)\ .$$

On multiplying relation (f) of (3.4.12) by $t(\sigma)$ as given by (3.4.11) and performing the integration with respect to σ in the interval $0 \leq \sigma \leq \epsilon$, we obtain,

c. $$\int_0^\epsilon t(\sigma)\ [\quad]\ \frac{dr}{d\sigma}\ .\ d\sigma = \frac{\epsilon}{4a}\ [1 + (\frac{a_1}{2} - \frac{13}{24})\epsilon + (\frac{a_2}{3} - \frac{17}{72}a_1 + \frac{7}{24})\epsilon^2$$

$$+ (\frac{a_3}{4} - \frac{7}{48}a_2 + \frac{11}{96}a_1 - \frac{75}{512})\epsilon^3$$

$$+ (\frac{a_4}{5} - \frac{5}{48}a_3 + \frac{21}{320}a_2 - \frac{103}{1920}a_1 + \frac{11159}{30720})\epsilon^4] + O(\epsilon^6)\ .$$

Therefore on equating the sum of (a) and (b) to (c) above and then equating corresponding powers of ϵ, the following four relationships for the unknown coefficients a_1, a_2, a_3 and a_4 emerge

(3.4.14)

a. $\frac{3}{2}a_1 - \frac{5}{4} = \frac{3}{4}$,

b. $(\frac{a_2}{3} - \frac{17}{72}a_1 + \frac{7}{24}) + (a_2 - \frac{17}{24}a_1 + \frac{13}{32}) = -\frac{15}{32}$,

c. $(\frac{a_3}{4} - \frac{7}{48}a_2 + \frac{11}{96}a_1 - \frac{75}{512}) + (a_3 - \frac{17}{24}a_2 + \frac{13}{32}a_1 - \frac{107}{512}) = \frac{61}{256}$,

d. $(\frac{a_4}{5} - \frac{5}{48}a_3 + \frac{21}{320}a_2 - \frac{103}{1920}a_1 + \frac{11159}{30790}) + (a_4 - \frac{17}{24}a_3 + \frac{13}{32}a_2 - \frac{107}{512}a_1 + \frac{145}{3072})$

$= -\frac{473}{5120}$.

From the above relations (a), (b), (c) and (d) we obtain respectively

$$a_1 = \frac{4}{3}, \quad a_2 = \frac{5}{72}, \quad a_3 = \frac{89}{120}, \quad a_4 = \frac{264419}{442368} .$$

The dependence of time on the bore locus on the parameter σ is then given by

(3.4.15) $$t(\sigma) = \frac{2}{3a}[1 + \frac{4}{3}\sigma + \frac{5}{72}\sigma^2 + \frac{89}{120}\sigma^3 + \frac{264419}{442368}\sigma^4 + O(\sigma^5)] .$$

The solution is completed by obtaining x as a function of σ. This is easily accomplished by using the relation $dx = \dot{\xi}\, dt$ on the bore locus. After some algebra, this procedure then yields the result

(3.4.16) $$x(\sigma) = \frac{2}{3a}[1 + \frac{4}{3}\sigma + \frac{41}{72}\sigma^2 + \frac{203}{240}\sigma^3 + \frac{953856}{552962}\sigma^4 + O(\sigma^5)] .$$

The expansions (3.4.15) and (3.4.16) provide a parametric representation for the path of the bore in the physical plane. The solution shows that the bore front starts at the point $(\frac{2}{3a}, \frac{2}{3a})$ in the (x, t)-plane with an initial velocity $\dot{\xi} = 1$ and an initial acceleration $\ddot{\xi} = \frac{213}{256}a$. As the bore gains in strength both its velocity and acceleration increase. On the basis of the 'simple wave' approximation of Friedrichs the acceleration of the bore is constant for all times t with a value as given above. By comparing (3.4.15) with (3.2.5)b, it is seen that the simple wave theory gives the first three terms of the expansion for t by the present method, that is up to the term in σ^2. The additional terms obtained above when taken in conjunction with the corresponding ones for x indicate that the rate at which the bore grows in intensity is slightly overestimated by the simple wave approximation.

Numerical results for t as a function of σ are given in Table 1 for the range $0 \leq \sigma < \frac{2}{3}$ and in Figure 16, the results are shown graphically. For values of σ less than $\frac{4}{9}$, the percentage difference in the results from the two theories is less than 5.2 and consequently within this range Friedrichs' theory shows very good agreement with the present results. However, as σ increases from this value, the terms of order σ^3 become increasingly more important and the results from the integral equation should be used.

Results analogous to (3.4.15) and (3.4.16) were obtained by Pillow (1949) for the formation of a shock. In fact the method used here is very similar to that of Pillow although, because of the absence of entropy variations, the integral equation (3.4.10) is exact whereas in Pillow's work the equivalent equation was derived on an approximation valid to terms of order σ^3. Thus Pillow does not continue the expansion beyond the term in σ^3 since further terms would not be significant. In relation (3.4.15) the coefficient of σ^3 is positive, indicating that the bore grows somewhat less rapidly than the simple wave approximation suggests. Pillow appears to have misinterpreted his results, stating that the shock grows more rapidly when determined from the higher approximation.

By a simple generalisation of the integral equation (3.4.7), the solution for the time t at any interior point of the curvilinear triangle MYX may be found to the same order of approximation as above. It will be remembered that on the basis of Friedrichs' theory this region is the continuation of the incident simple wave domain.

In Figure 17, let K be the point having co-ordinates r_i and s_j with $i \neq j$, that is, the point K does not lie on the curve $P(r, s) = 0$. The lines $r = r_i$ and $s = s_j$ cut $P(r, s) = 0$ at the unique points I and J respectively. On the arc JI, the time t is a known function of σ and thus of $\frac{2}{3}(\frac{3}{2}r - \frac{1}{2}s - 1)$ and also at all points of the arc the derivative $\frac{ds}{dr}$ is finite and non-zero. Consequently we may apply Riemann's method to obtain the required value of t at K. In fact, from (3.4.7) we find immediately,

$$(3.4.17)\qquad t(K) = [\{\tfrac{3}{2}r - \tfrac{1}{2}s - \dot{\xi}(r,s)\}\,wt](I) - [\{\tfrac{1}{2}r - \tfrac{3}{2}s - \dot{\xi}(r,s)\}\,wt](J)$$

$$+\int_{r_j}^{r_i} t[w(1-\frac{ds}{dr}) - \frac{d}{dr}\{(r-s-\dot{\xi})w\} - \frac{r+s}{2}(w_r - w_s\frac{ds}{dr})]\,dr\ .$$

All the quantities on the R.H.S. of this relation are known in terms of r_i and r_j. Consequently on using the relation $P(r_j, s_j) = 0$, $t(K)$ can be found solely in terms of the point co-ordinates r_i and s_j. The corresponding solution for $x(K)$ can then be obtained by a method similar to that above or by using relation (3.1.1). Equation (3.4.17) is valid for any point K of the region bounded by the bore locus, the C^- characteristic through the initial point of formation of the bore, M, and the reflection of this characteristic at the piston path.

Although we are concerned mainly with the initial stages of the process of formation we can describe quickly the remainder of the flow-field when the motion of the piston is such that the acceleration ceases before the point where the backward propagated C^- characteristic from M reaches the piston path at the point B, as represented in Figure 18. C is the point at which the acceleration of the piston ceases.

The solution in the region MHR has already been discussed and is unaffected by the motion of the piston beyond C. The subsequent behaviour of the flow can be determined from Figure 18 and is as follows.

We suppose that the piston continues in the water for all times $t_p > t_p^x$ with constant velocity $u^x = at_p^x$. The co-ordinates of C are taken as x_p^x, t_p^x. On CB, (r-s) is constant and as s has the value unity in the simple wave domain OMB, it is evident that in CHB both r and s are constant whilst in HRM_1B, r only is constant. The flow in this latter region is thus a backward-facing simple wave. Since s decreases from unity as the strength of the bore increases, it can be shown, by an argument similar to that given in §3 Chapter II, that the simple wave is a compression wave and we assume no secondary bores form.

In the region RM_1V, r is constant and consequently the velocity of the bore front does not alter in the interval RM_1. Therefore the value of s propagated back into the water along the C^- characteristics is constant. Hence we deduce:

(i) The flow in RM_1V is of constant properties.

(ii) The region $M_1VC_1H_1$ is a forward-facing simple wave domain.

Along the section of the piston path BC_1, u is constant although r and s vary. However, as s is decreasing it follows that r also must decrease as we move along the piston path in the direction of increasing time. We can then show, as above, that the simple wave in $M_1VC_1H_1$ is a rarefaction wave and consequently the velocity of the bore in the interval M_1R_1 will diminish.

Finally, the region $H_1B_1C_1$ is one in which s is constant and hence r also is constant, that is $H_1B_1C_1$ is a region of uniform properties.

The whole cycle of flow domains is then repeated exactly as given above until the velocity of the particles of water behind the bore is that of the piston, that is u^x.

Physically the motion represents a group of pressure waves undergoing successive compression and expansion as it is reflected off the bore front and the piston path.

The change in the Riemann invariants r and s together with the change in the velocity of the bore between the points R_1 and R can be estimated as follows.

By using the properties of the C^+ and C^- characteristics, we can write

$$r(R_1) - r(M_1) = r(C_1) - r(B) \quad .$$

However, as u is constant on BC_1

$$r(C_1) - r(B) = s(C_1) - s(B)$$

$$= s(R) - s(M) \quad ,$$

and hence

$$r(R_1) - r(M_1) = s(R) - s(M) \; .$$

The value of s at M, that is $s(M)$, is unity and consequently when the above relation is expressed in terms of the parameter σ from relations (3.1.5), we find

$$(3.4.18) \qquad r(R_1) - r(M_1) = -\frac{3}{64}\sigma^3(R) + O\{\sigma^4(R)\} = r(R_1) - r(R) \; ,$$

since region M_1VR is one of constant properties. By substituting for $r(R_1)$ and $r(R)$ in terms of $\sigma(R_1)$ and $\sigma(R)$, we obtain from (3.1.5),

$$\sigma(R_1) - \frac{1}{64}\sigma^3(R_1) + O\{\sigma^4(R_1)\} = \sigma(R) - \frac{1}{16}\sigma^3(R) + O\{\sigma^4(R)\} \; ,$$

and consequently for $\sigma(R_1)$, $\sigma(R)$ sufficiently small the following series expansion is justifiable,

$$\sigma(R_1) = \sigma(R) - \frac{3}{64}\sigma^3(R) + O\{\sigma^4(R)\} \quad .$$

Using the above relation together with (3.1.5), the change in s from R_1 to R can then be found and is given by

$$(3.4.19) \qquad s(R_1) - s(R) = \frac{27}{4096}\sigma^5(R) + O\{\sigma^6(R)\} \quad ,$$

and in a similar manner, the change in the velocity of the bore in the same interval, can be obtained and is

$$(3.4.20) \qquad \dot{\xi}(R_1) - \dot{\xi}(R) = -\frac{9}{256}\sigma^3(R) + O\{\sigma^4(R)\} \quad .$$

It is interesting to note that on the simple wave approximation, the results corresponding to the above are

$$r(R_1) - r(R) = 0 \ ,$$

$$s(R_1) - s(R) = 0$$

and $$\dot{\xi}(R_1) - \dot{\xi}(R) = 0 \ ,$$

and although terms of order $\sigma^3(R)$ are neglected in this theory, the results are much better than would be expected owing to the small magnitudes of the coefficients of $\sigma^3(R)$ and $\sigma^5(R)$ in relations (3.4.18), (3.4.19) and (3.4.20).

§5. COMPLETE 3RD ORDER SOLUTION FOR THE DECAY OF A BORE

We quickly recapitulate the main points of the statement of this problem. A piston is pushed with uniform velocity u_1 into the still water causing a bore of constant strength to advance in the stagnation region. After a finite time, the piston is retracted from the water with a constant velocity $u_2 = -2s_n$, thus generating a simple rarefaction wave, point-centred at the origin, which is just completed by vacuum conditions. The motion is thus expansive everywhere behind the bore front and no secondary shocks form.

The flow is complicated by the effects due to the disturbances which are propagated along the C^- characteristics from the bore back into the fluid and which in turn interact with the oncoming C^+ characteristics and so affect the rate of decay of the bore. The method of solution which is used takes into account the effect of this interaction.

In the physical plane, the flow-field is as shown in Figure 12 and the region of interest, consisting of the pencil of C^+ characteristics which contribute to the process of decay, is bounded by the C^+ characteristic of the simple wave at the bore and by the locus of the bore itself. In the characteristic plane this region is transformed into the finite and bounded domain NY_1M, as shown in Figure 15.

As before, the system is governed by the Euler-Poisson-Darboux equation, (3.1.2), for the time t as a function of r and s. The boundary conditions appropriate to this problem are supplied by (3.1.9) and (3.1.13), that is

$$(3.5.1) \qquad \text{(i) On } F(r,s) = 0 \quad , \qquad t_s = G(r,s)\, t_r \quad ,$$

$$\text{(ii) On } s = s_n \quad , \qquad t = t_n \left(\frac{r + s_n}{r_n + s_n}\right)^{-\frac{3}{2}} \quad .$$

The Riemann function satisfying the adjoint equation of (3.1.2) is similar to that defined in the previous section. However, a typical field point in the present application now lies on the line $s = s_n$ instead of $s = 1$. Accordingly in the present section we use

$$(3.5.2) \qquad w(r, s ; r_\theta, s_n) = \frac{(r+s_n)^{\frac{1}{2}}(r_\theta + s)^{\frac{1}{2}}}{(r_\theta + s_n)^{2}} F(-\tfrac{1}{2}, -\tfrac{1}{2} ; 1 , p) \quad ,$$

where $$p = \frac{(r-r_\theta).(s-s_n)}{(r+s_n).(r_\theta+s)} \quad .$$

This function is $(r+s)^{-1}$ times the classical Riemann function with field point (r_θ, s_n). The requisite proof is given in §D Appendix I. As before, s_θ is the s-co-ordinate of the point on the bore locus whose r-co-ordinate is r_θ. This is the point X in Figure 15. The procedure followed to solve this boundary value problem is similar in type to that given for the formation problem. We use Riemann's method to derive an integral equation for the time $t(r, s)$ on the bore locus. There is, however, one main difference between the integral equation in this case and that of the previous section which is that now the equation has a singularity as the upper limit of integration r_θ tends to unity, the value assumed in the limit as the bore tends to a 'sound wave' in a stagnation region. This behaviour is expected as we now wish to consider the whole history of the bore and must therefore seek a solution which is valid for all $t \geq t_n$ and we would expect the bore to decay to zero strength only after an infinite time. Unfortunately an exact solution of the integral equation cannot be obtained owing to the complexity of the integrand. However, if as before the series representations of the bore relations, (3.1.5), in terms of σ are employed then an approximate solution can be obtained which is valid so long as the initial strength of the bore is such that the fourth power of the pressure difference across it can be neglected. The solution is thus limited to bores which are not too strong.

However, the method does yield the complete history of the bore within this range of pressure difference.

The integral equation is derived in exactly the same manner as (3.4.7) and is indeed identical with it except that M is now replaced by N and the lower limit of the integral is r_n instead of 1. When use is made of (3.5.1) then the form of the integral equation corresponding to (3.4.9) is

(3.5.3)

$$[\{\frac{3}{2}r_\theta - \frac{1}{2}s_\theta - \dot{\xi}(r_\theta, s_\theta)\} (r_\theta + s_n)^{-\frac{3}{2}} (r_\theta + s_\theta)^{\frac{1}{2}}] \, t(r_\theta, s_\theta)$$

$$= t_n[\{\frac{3}{2}r_n - \frac{1}{2}s_n - \dot{\xi}(r_n, s_n)\} (r_n + s_n)^{\frac{1}{2}} (r_\theta + s_n)^{-\frac{3}{2}}]$$

$$- \int_{r_n}^{r_\theta} t(r, s)[w\frac{d\dot{\xi}(r, s)}{dr} - \{(\frac{3}{2}r - \frac{1}{2}s - \dot{\xi})w_r + (\frac{1}{2}r - \frac{3}{2}s - \dot{\xi})w_s\frac{ds}{dr}\}] \, dr \; .$$

Mathematically the most significant difference between this equation and (3.4.9) is that the term on the R.H.S. other than the integral does not vanish when $r_\theta = 1$ as it does in (3.4.9). Since the coefficient of $t(r_\theta, s_\theta)$ on the L.H.S. is zero when $r_\theta = 1$ in both cases this explains why we get an infinite value of t when $r_\theta = 1$ in (3.5.3) although not in (3.4.9).

Before deriving the parametric solution for x and t on the bore locus we can obtain from (3.5.3) the initial rate of change of time on the bore locus with respect to r_θ. This expansion is exact and is consequently valid for a bore of any initial strength. We differentiate (3.5.3) with respect to r_θ and then take the derivative at the initial point of modification of the bore, where $r_\theta = r_n$, $s_\theta = s_n$ and obtain immediately,

(3.5.4)

$$\Big\{ \Big[\frac{3}{2} - \frac{1}{2}\frac{ds_\theta}{dr_\theta} - \frac{d\dot{\xi}}{dr_\theta}\Big]_n (r_n + s_n)^{-1} + \frac{1}{2}\Big(\frac{3}{2}r_n - \frac{1}{2}s_n - \dot{\xi}(n)\Big)(r_n + s_n)^{-2}\Big(1 + \frac{ds_\theta}{dr_\theta}\Big)_n$$

$$- \frac{3}{2}\Big(\frac{3}{2}r_n - \frac{1}{2}s_n - \dot{\xi}(n)\Big)(r_n + s_n)^{-2} + \Big[\frac{3}{2}r_n - \frac{1}{2}s_n - \quad (n)\Big](r_n + s_n)\frac{1}{t_n}\Big(\frac{dt}{dr_\theta}\Big)_n \Big\}\ t_n$$

$$= -\frac{3}{2}\Big[\frac{3}{2}r_n - \frac{1}{2}s_n - \dot{\xi}(n)\Big](r_n + s_n)^{-2}\ t_n - \Big\{\Big[w(r_n, r_n, s_n, s_n)\Big(\frac{d\dot{\xi}}{dr_\theta}\Big)_n\Big]$$

$$- \Big[\frac{3}{2}r_n - \frac{1}{2}s_n - \dot{\xi}(n)\Big]\Big[w_r(r_\theta, r, s, s_n)\Big]_{\substack{r=r_\theta=r_n \\ s=s_\theta=s_n}}$$

$$- \Big[\frac{1}{2}r_n - \frac{3}{2}s_n - \dot{\xi}(n)\Big]\Big[w_s(r_\theta, r, s, s_n)_{\substack{r=r_\theta=r_n \\ s=s_\theta=s_n}}\Big(\frac{ds}{dr}\Big)_n\Big]\ t_n \quad .$$

In the above, the integral of the derivative with respect to r of the integrand is zero as the integrand is bounded at the point N and the range of integration is zero.

From (3.5.2) it is easily seen that:

$$\text{(i)}\quad w(r_n, r_n, s_n, s_n) = (r_n + s_n)^{-1}$$

$$\text{(ii)}\quad \{w_r(r_\theta, s_n, r, s)\}_{\substack{r=r_\theta=r_n \\ s=s_\theta=s_n}} = \frac{(r_n+s_n)^{-2}}{2} = \{w_s(r_\theta, s_n, r, s)\}_{\substack{r=r_\theta=r_n \\ s=s_\theta=s_n}} .$$

On substituting for the relevant quantities from the above relations into (3.5.4) and performing the necessary reductions, we finally obtain the simple result,

$$(3.5.5)\qquad \frac{1}{t_n}\left\{\frac{dt(r_\theta, s_\theta)}{dr_\theta}\right\}_n = -\frac{3}{2}\left\{\frac{3}{2}r_n - \frac{1}{2}s_n - \dot{\xi}(r_n, s_n)\right\}^{-1} .$$

For moderately small values of the parameter σ_n, the above expression can be expanded by using (3.1.5) to give

$$(3.5.6)\qquad \left(\frac{dt}{d\sigma}\right)_n = -\frac{2t_n}{\sigma_n}\left[1 + \frac{5}{24}\sigma_n - \frac{1}{18}\sigma_n^2 + \frac{149}{13824}\sigma_n^3 + \frac{2839}{331776}\sigma_n^4 + O(\sigma_n^5)\right] .$$

We may proceed in a like manner and derive the value at the initial point N of any of the succeeding derivatives of t with respect to r_θ. By a simple application of Leibnitz' differential theorem it can then be seen that the m th derivative, for example, has a leading term of order $\left(\frac{1}{\sigma_n}\right)^m$ when expanded as a power series in σ_n. With a knowledge of these derivatives, the corresponding derivatives of x with respect to r_θ could then be obtained by using the relation $dx = \dot{\xi}\, dt$, which is valid on the bore locus. In this way a Taylor series expansion for x and t in terms of r and r_n could be generated. This solution would hold for bores of any strength but it would be valid only in the neighbourhood of the initial point of modification of the bore. We do not derive this solution here as in §6 of this chapter a more general method, based on Meyer's 'focussing equations' (Meyer and Mahony, 1956), is given. We do, however, derive a series representation for x and t on the bore locus which is applicable to bores of moderate strength and which gives the complete history of the bore locus. Relation (3.5.6) will be useful for providing a means of estimating the range of validity of this solution.

It is convenient to define a new dependent variable, T, by

$$(3.5.7) \qquad T(r_\theta, s_\theta) = \left[\frac{3r_\theta - s_\theta - 2\,(r_\theta, s_\theta)}{3r_n - s_n - 2\,(r_n, s_n)}\right] \frac{(r_\theta + s_\theta)^{\frac{1}{2}}}{(r_n + s_n)^{\frac{1}{2}}} (r_\theta + s_n)^{-\frac{3}{2}} \frac{t(r_\theta, s_\theta)}{t_n} .$$

In terms of T, the integral equation (3.5.3) then becomes

$$(3.5.8) \qquad T(r_\theta, s_\theta) = (r_\theta + s_n)^{-\frac{3}{2}} - \int_{r_n}^{r_\theta} T(r, s) \frac{(r+s)^{-\frac{1}{2}} (r+s_n)^{\frac{3}{2}}}{(\frac{3}{2}r - \frac{1}{2}s - \dot{\xi}(r, s))} k_1(r_\theta, s_n, r, s)\, dr ,$$

where

$$k_1(r_\theta, s_n, r, s) = [w\frac{d\dot{\xi}}{dr} - \{(\frac{3}{2}r - \frac{1}{2}s - \dot{\xi})w_r + (\frac{1}{2}r - \frac{3}{2}s - \dot{\xi})w_s\frac{ds}{dr}\}] .$$

Inspection of $k_1(r_\theta, s_n, r, s)$ shows that it is a bounded function in the range $1 \leq r \leq r_\theta \leq r_n$ and hence that the singularity in the integral equation (3.5.8) arises only from the term $(\frac{3}{2}r - \frac{1}{2}s - \dot{\xi})$ in the denominator of the integrand. Consequently, we would expect to get a uniformly valid approximate solution by expanding $k_1(r_\theta, s_n, r, s)$ as a double power series in $(r-1)$ and $(r_\theta - 1)$, that is in σ and ϵ, where ϵ is the value of σ at the point X in Figure 15. The kernal of (3.5.8) can be expressed in terms of s_n, σ and ϵ to any required degree of accuracy. However, as the order of the expansion is increased the algebraic detail becomes increasingly more cumbersome. Consequently, to illustrate the method employed we expand the kernel to terms of order σ^3 only. To this order, we may replace s_n by unity wherever it occurs with terms of order σ. After a considerable amount of algebra, the details of which are given in §C Appendix I, the required expansion for $k_1(\sigma, \epsilon)$ is obtained as

$$k_1(\sigma, \epsilon)\frac{dr}{d\sigma} = \frac{3}{2\sqrt{2}}(1+s_n)^{-\frac{3}{2}}[1 + \frac{5\sigma - 9\epsilon}{12} - \frac{2\sigma^2 + 10\sigma\epsilon - 15\epsilon^2}{32} - \frac{22\sigma^3 - 30\sigma^2\epsilon - 50\sigma\epsilon^2 + 67\epsilon^3}{256} + O(\sigma^4)] .$$

The remaining terms contributing to the kernel of (3.5.8) are entirely functions of σ and may be shown to be given by

$$(3.5.9) \qquad \frac{(r+s)^{-\frac{1}{2}}(r+s_n)^{\frac{3}{2}}}{\frac{3}{2}r-\frac{1}{2}s-\dot{\xi}} = \frac{2\sqrt{2}}{3}\,\frac{(1+s_n)^{\frac{3}{2}}}{\sigma}\,[1+\frac{17}{24}\sigma+\frac{55}{576}\sigma^2-\frac{77}{6912}\sigma^3+O(\sigma^4)] \ .$$

The proof of this representation is given in §C Appendix I. The method adopted to solve the integral equation (3.5.8) is to transform it to an equivalent ordinary differential equation for $T(\epsilon)$. To this end, we require to write $k_1(\sigma,\epsilon)$ as far as possible in a Pincherle-Goursat form. The form which is most suitable is

$$(3.5.10) \qquad k_1(\sigma,\epsilon)\frac{dr}{d\sigma} = \frac{3}{2\sqrt{2}}(1+s_n)^{-\frac{3}{2}}[1-\frac{3}{4}\epsilon+\frac{15}{32}\epsilon^2-\frac{67}{256}\epsilon^3+O(\epsilon^4)] \times$$

$$[1+\frac{5}{12}\sigma-\frac{1}{16}\sigma^2-\frac{11}{128}\sigma^3+\frac{9}{128}\epsilon\sigma^2+O(\sigma^4)] \ .$$

Thus by multiplying (3.5.9) and (3.5.10) the kernel $k(\sigma,\epsilon)$ of the integral equation (3.5.8), when expressed as a double power series in σ and ϵ is to the order σ^3, given by

$$(3.5.11) \qquad k(\sigma,\epsilon) = \frac{1}{\sigma}[1-\frac{3}{4}\epsilon+\frac{15}{32}\epsilon^2-\frac{67}{256}\epsilon^3+O(\epsilon^4)]$$

$$[1+\frac{9}{8}\sigma+\frac{21}{64}\sigma^2-\frac{702}{6912}\sigma^3+\frac{9}{128}\epsilon\sigma^2+O(\sigma^4)] \ .$$

If we now define

$$(3.5.12) \qquad M(\epsilon) = (r_0+s_n)^{\frac{3}{2}}\,T(\epsilon) \ ,$$

then the integral equation (3.5.8) may be written after some algebra, as

$$(3.5.13) \qquad M(\epsilon) = 1 - \int_{\sigma_n}^{\epsilon} \frac{M(\sigma)}{\sigma}\left[1+\frac{3}{8}\sigma - \frac{3}{64}\sigma^2 - \frac{21}{256}\sigma^3 + \frac{9}{128}\epsilon\,\sigma^2 + O(\sigma^4)\right] d\sigma \quad .$$

The proof of this result is given in §E, Appendix I.

By a double differentiation of (3.5.13) with respect to ϵ, we find that the integral equation to this approximation is equivalent to the following second order linear ordinary differential equation,

$$(3.5.14) \qquad \epsilon^2 \frac{d^2M}{d\epsilon^2} + \left\{1+\frac{3}{8}\epsilon - \frac{3}{64}\epsilon^2 - \frac{3}{256}\epsilon^3 + O(\epsilon^4)\right\}\epsilon\frac{dM}{d\epsilon}$$

$$- \left\{1+\frac{3}{64}\epsilon^2 - \frac{15}{128}\epsilon^3 + O(\epsilon^4)\right\} M = 0 \quad ,$$

with boundary conditions

(i) $M(\sigma_n) = 1$,

(ii) $M'(\sigma_n) = -\frac{1}{\sigma_n}\left[1+\frac{3}{8}\sigma_n - \frac{3}{64}\sigma_n^2 - \frac{3}{256}\sigma_n^3 + O(\sigma_n^4)\right]$.

Equation (3.5.14) has a regular singularity at $\epsilon = 0$ and consequently it can be solved by the method of Frobenius. The details of the situation are given in §E, Appendix I. The indicial equation gives exponents ± 1 at the origin but although these differ by an integer, the resulting series solution can in fact be generated without recourse to logarithmic singularities. When the solution is carried out to the appropriate degree of approximation and the initial conditions fitted in, we obtain

$$(3.5.15) \qquad M(\epsilon) = \frac{\sigma_n}{\epsilon}\Big[1 + \frac{3}{8}(\sigma_n - \epsilon) + \frac{3}{64}(\sigma_n^{\,2} - 3\sigma_n\epsilon + 2\epsilon^2)$$

$$- \frac{1}{1024}(37\sigma_n^{\,3} + 18\sigma_n^{\,2}\epsilon - 135\sigma_n\epsilon^2 + 80\epsilon^3) + O(\sigma_n^{\,4})\Big] .$$

It might be objected that the nature of the singularity in this solution would vary according as the order of the expansion of the integrand of (3.5.13). However, this is not the case. If, for example, we consider the expansion to terms of order m in ϵ then the integral equation corresponding to (3.5.13) would be of the form

$$M(\epsilon) = 1 - \int_{\sigma_n}^{\epsilon} \frac{M(\sigma)}{\sigma}\Big[\sum_{i=0}^{m} f_i(\sigma)\,\epsilon^i\Big]\,d\sigma \quad ,$$

in which the f_i are functions of σ and are known.

This integral equation may be transformed into the equivalent ordinary differential equation by differentiating $(m+1)$ times with respect to ϵ. On examination of the indicial equation it is then seen that the exponents are $\pm 1, 2, 3, 4, \ldots, m$. That is, the nature of the singularity given by (3.5.15) is correct on the basis of the approximation introduced by expansion of the kernal of the integral equation. The remaining exponents, $1, 2, 3, \ldots, m$ serve to generate the higher order terms in ϵ and σ_n.

The solution for t on the bore locus as a function of σ, σ_n is now readily obtained by using the relations (3.5.12) and (3.5.7). The details of the solution are given in §E, Appendix I and the required result is found to be

$$(3.5.16) \qquad \frac{t(\sigma,\sigma_n)}{t_n} = \frac{\sigma_n^2}{\sigma^2}\Big[1 + \frac{5}{12}(\sigma_n - \sigma) + \frac{1}{288}(9\sigma_n^2 - 50\sigma_n\sigma + 41\sigma^2)$$

$$- \frac{1}{27648}(999\sigma_n^3 + 360\sigma_n^2\sigma - 4313\sigma_n\sigma^2 + 2954\sigma^3) + O(\sigma_n^4)\Big]$$

in which ϵ has been rewritten as σ.

The corresponding solution for $x(\sigma,\sigma_n)$ can now be obtained by using the bore relation $dx = \dot{\xi}(\sigma)dt$. However, we do not give it explicitly here.

Examination of (3.5.16) shows that it has the anticipated behaviour near $\sigma = 0$. As will be shown later, it is also accurate near $\sigma = \sigma_n$ during the initial stages of decay of the bore. The corresponding result obtained by Friedrichs' simple wave approximation, (3.2.5)a, agrees with the above result up to the linear term in the square bracket. The simple wave solution could be obtained from (3.5.13) by neglecting terms of order σ^2 in the integrand. The present approach leads to an approximation two degrees better than that of the simple wave although the initial assumption only involved retaining one additional power of σ. The method of solution, however, now takes into account the cumulative effects of the error introduced by the approximation.

The initial rate of decay of the bore is determined from the value of $\frac{dt}{d\sigma}$ at the point N where $\sigma = \sigma_n$. From (3.5.16), we obtain on differentiating with respect to σ,

$$(3.5.17) \qquad \left(\frac{dt}{d\sigma}\right)_n = -\frac{2t_n}{\sigma_n}\left[1 + \frac{5}{24}\sigma_n - \frac{1}{18}\sigma_n^{\ 2} + \frac{149}{13824}\sigma_n^{\ 3} + O(\sigma_n^{\ 4})\right] .$$

This expansion agrees to the term in $\sigma_n^{\ 3}$ with that obtained by expanding the exact result given by (3.5.5), that is (3.5.6). From (3.5.6) it is seen that the error term in the above solution is of order $10^{-2}\sigma_n^{\ 4}$ and it is not unreasonable to suppose that (3.5.16) will give an adequate description of the whole history of the bore whenever this expression can be neglected. The flow-field is represented in Figure 12 and as the simple wave is just completed by vacuum conditions, the C^- characteristics tend asymptotically to the piston path OB. The above solution for the locus of the bore, (3.5.16), determines the whole history of the bore within the approximation observed. By a similar method to that used in the formation problem, the flow can be determined at any interior point of the region bounded by the C^+ characteristic Y_1M on which $r = 1$, the portion of the C^- characteristic bounding the incident simple wave domain Y_1N and the locus of the bore NSM. In fact, on using a symbolism similar to that employed previously and with the appropriate boundary conditions for t, we may write down the following expression for the time t at any point $K(r_i, s_j)$ in this domain.

$$(3.5.18) \qquad t(K) = [\{\frac{3}{2}r - \frac{1}{2}s - \dot{\xi}(r,s)\}\,wt](I) - [\{\frac{1}{2}r - \frac{3}{2}s - \dot{\xi}(r,s)\}\,wt](J)$$

$$+ \int_{r_j}^{r_i} t[w\frac{d\dot{\xi}}{dr} - \{(\frac{3}{2}r - \frac{1}{2}s - \dot{\xi})w_r + (\frac{1}{2}r - \frac{3}{2}s - \dot{\xi})\frac{ds}{dr}w_s\}]\,dr \; .$$

The corresponding result for the value of x at the point K can then be obtained by using the relations (3.1.1). The flow in the remaining portion of the physical plane BOY_1M has been shown in §3 Chapter II to be everywhere non-compressive. It can now be determined explicitly by solving the boundary value problem in which t satisfies the Euler-Poisson-Darboux equation, (3.1.2), with the boundary conditions provided by the simple wave relationships on the C^- characteristic Y_1B and the known variation of t with respect to s on the C^+ characteristic on which $r = 1$, that is Y_1M, determined from (3.5.18).

§6. SOME NUMERICAL RESULTS FOR THE DECAY OF A BORE

We now make a comparison of numerical results drawn from the present theory and Friedrichs' simple wave theory. For convenience, the section is sub-divided into three parts.

(i) Table 2 shows the values of $(\frac{dt}{dr})_n$ on the locus of the bore as calculated by three different formulae for a range of values of c_n^2, the square of the sound velocity behind the bore immediately before the interaction takes place. In the following work c_n is such that $1.2 \leq c_n^2 \leq 3$ corresponding to values of σ_n, $0.1913 \leq \sigma_n \leq 1.5767$. The three cases are

(a) The exact value from the integral equation, given by (3.5.5).

(b) The value from the series solution from the integral equation as determined from (3.5.17) and (3.1.5).

(c) The value determined from the simple wave theory and given by (3.2.5)a with (3.1.5).

Examination of Table 2 indicates that as expected the approximate solution determined from the integral equation is more accurate than that determined from the simple wave theory. However, both approximations are remarkably good when compared with the exact values except where the bore is initially very strong. The columns showing the percentage difference in the values of $-(\frac{dt}{dr})_n$ from either approximate theories relative to the exact value, given by (3.5.5), indicate, however, that even when the bore is initially weak the results from the present approximate method are significantly better than those from the simple wave approximation suggesting that the terms of order σ_n^2 make a significant contribution to the final numerical value. The two approximate solutions are applicable strictly only to bores for which $\sigma_n < \frac{2}{3}$, that is $c_n < 1.6$, as the expansions of the bore relations are convergent for such values of σ. Nevertheless, the table would seem to indicate the approximate solution for $-(\frac{dt}{dr})_n$ as a function of σ and σ_n obtained from the integral equation could be extended to bores for which σ may be as high as 1.57, corresponding to $c_n^2 = 3$ as the percentage difference is of the order of only 1.4%. For the same degree of error, the simple wave theory, however, ought to be limited to bores for which the value of $\sigma_n < \frac{2}{3}$.

(ii) The second sub-division is concerned with the results of the simple wave approximation. The significant solution of the decay problem by the simple wave is given by (3.2.5)a. However, in Chapter II it was mentioned that Friedrichs derives a closed analytic form for the solution which in fact includes terms of the same order as are neglected by the hypothesis of the theory. Lax (1948) applied the simple wave approximation to the decay of bores and employs the closed form of the solution to derive numerical results which are given in graphical form. It will be interesting to compare some numerical values of $\frac{t}{t_n}$ as obtained from the two solutions.

In §F Appendix I, the closed analytic form of Friedrichs' solution for t as a function of σ and σ_n is derived and we shall merely quote here the result.

$$\frac{t}{t_n} = \left(\frac{\sigma_n}{\sigma}\right)^2 \left[\frac{1 - \frac{5}{24}\sigma}{1 - \frac{5}{24}\sigma_n}\right]^2 \quad . \tag{3.6.1}$$

The significant solution is derived in §3 of this chapter and is

$$\frac{t}{t_n} = \left(\frac{\sigma_n}{\sigma_n}\right)^2 \left[1 + \frac{5}{12}(\sigma_n - \sigma) + O(\sigma_n^{\,2})\right] \quad . \tag{3.6.2}$$

We compare numerical results for $\frac{t}{t_n}$ from the above expressions for a range of values of σ and σ_n. The desired numerical values are listed in Table 3 and it has been found convenient to use c and c_n as independent variables instead of σ and σ_n. This change is made merely as it is felt that c is more meaningful physically than the parameter σ. The main conclusion to be drawn from Table 3 is that the values of $\frac{t}{t_n}$ when calculated from the closed expression, (3.6.1), are consistently underestimated relative to those calculated from the significant result, (3.6.2). This means that the actual decay of the bore is thus slightly overestimated when (3.6.1) is regarded as the solution from the simple wave hypothesis. Also, as the initial strength of the bore increases, that is as c_n increases, the magnitude of the overestimation increases correspondingly. For values of t in the neighbourhood of the initial point of interaction, t_n, the two forms predict values whose magnitudes differ only by 1% and as such are not actually significant. However for larger values of t this difference becomes much more pronounced so that caution must be exercised when claiming any significance for the results from (3.6.1);

for example, when $c_n^2 = 1.5$ this difference is of the order of 4 % with $c^2 = 1.05$ and for $c_n^2 = 1.8$, the difference is of order 10 % corresponding to $c^2 = 1.08$.

It seems appropriate to note that Lax (1948) suggests a numerical method which would be applicable to the decay of very strong bores. The proposed technique consists of two stages. Firstly, a finite difference scheme could be used to obtain numerical values when the bore is very strong. Secondly, to use the simple wave approximation when the bore has decayed sufficiently. It is questionable that this method would yield valid results since it would seem to assume that the flow from the incident simple wave remains unmodified after interaction with a strong bore. Even though the region in which the bore is strong is very small, the changes propagated backwards along the C^- characteristics will affect the oncoming C^+ characteristics appreciably. Hence, although the bore becomes weak after a sufficient time has elapsed and the flow in the region behind it approximates in character to a simple wave, this 'second' wave will not be the continuation of the incident one which Lax would seem to assume.

(iii) The last sub-division is concerned with a comparison of values of $\frac{t}{t_n}$ as a function of c^2, as obtained from the present theory and the significant form of Friedrichs' theory. By this means it is possible to give an indication of the range of values of c_n for which the simple wave approximation is no longer accurate. The results are contained in Table 4 for a range of values of c^2 and c_n^2. These results ought strictly to be confined to values of $\sigma_n < \frac{2}{3}$, that is $c_n^2 < 1.6$. However, in view of the close agreement with the exact solution obtained by the present method for values of $\left(\frac{dt}{dr}\right)_n$, shown in Table 2, it is not unreasonable to propose a 'range of application' given by $1 \leq c_n^2 < 3$, for the present theory. Examination of Table 4 in the range of values of c_n, $1 \leq c_n^2 \leq 1.8$, which corresponds to 'weak' bores, indicates that the two theories predict values of $\frac{t}{t_n}$ which are in good agreement although the simple wave approximation consistently underestimates the value of $\frac{t}{t_n}$ given by the present method. The maximum percentage difference is 10 % when $c_n^2 = 1.8$. This is to be expected as the term of order σ^2 in the present solution, (3.5.16), is positive for all values of σ and σ_n. Friedrichs' theory therefore overestimates the rate at which the bore decays for this range of initial bore strength. For bores of greater initial strength, the difference in the results becomes more pronounced, being 10 % when $c_n^2 = 1.8$ and 50 % when $c_n^2 = 3$. The simple wave approximation is not accurate in this range of bore strength as terms of order σ_n^2 will become increasingly important. The present approximate method should provide a reasonable indication of the order of magnitude of $\frac{t}{t_n}$ in this range. As the initial strength of the bore increases yet further, caution must be exercised in claiming any precise significance of the results from the present method as terms of order σ_n^4 will now become important.

In conclusion, the simple wave theory gives a reasonably accurate picture of the flow-field so long as the initial strength of the bore is such that $c_n^2 < 3$, the solution developed by the present method can be expected to describe the flow-field more adequately.

§7. AN ALTERNATIVE METHOD FOR THE DECAY OF BORES

The discussion on phenomena associated with bores is concluded by presenting a method for the initial stages of decay of a bore whose strength is not limited. We could of course derive such a solution from the integral equation by the following two methods.

(i) We may proceed as in the previous section and obtain the succeeding derivatives of t with respect to r_0 at the initial point of modification of the bore and thus generate a Taylor series expansion for time t on the bore.

(ii) As in the formation problem, a power series representation for t on the bore, of the form

$$t = \sum_i a_i(\sigma_n - \sigma)^i$$

could be substituted into the integral equation (3.5.3) and the requisite coefficients a_i determined.

Either of these methods would be useful for describing the early stages of the process of decay but could not be used for large values of t since the respective series would become divergent as $\sigma \to o$. Rather than use the above methods, however, we shall obtain the solution for the flow-field in the early stages of decay by a method introduced by Meyer (1956). This method too cannot be used for large values of t but it has three advantages over either of the above methods. First, it does not employ the Riemann function at all and hence avoids complicated expansions of the hypergeometric function. Secondly, it can be used to describe the entire flow behind the bore in a more tractable form than the solution given by the integral equation, (3.5.18). Lastly, the extension of this method to the initial stages of decay of bores caused by a more general piston motion can be quickly obtained. This latter advantage is relatively important since we consider only the initial stages of decay and consequently the effects due to secondary bores, which in general will be formed in this case, can be neglected.

The focussing equations are derived from the Euler-Poisson-Darboux equation in the dependent variable, time t, (3.1.2). If $U(r, s)$, $V(r, s)$ are functions defined by

$$(3.7.1) \qquad U(r,s) = \left(\frac{r+s}{r_n+s_n}\right)^{\frac{3}{2}} t_r(r,s) \ , \quad V(r,s) = \left(\frac{r+s}{r_n+s_n}\right)^{\frac{3}{2}} t_s(r,s) \ ,$$

then (3.1.2) may be replaced by two interdependent first order partial differential equations, the focussing equations,

$$(3.7.2) \qquad (r+s)U_s + \frac{3}{2}V = 0 \quad , \quad (r+s)V_r + \frac{3}{2}U = 0 \ .$$

We assume that the piston is pushed with uniform velocity u_1 into the still water and is then withdrawn, after a finite time, in an arbitrary manner. The origin of the co-ordinate system is chosen as the point on the piston path where the constant motion of the piston ceases. The parametric representation of the piston path is given in terms of two functions X and T of the Riemann invariant r which is assumed to be continuous and differentiable as required. The flow-field in the physical and characteristic planes is as illustrated in Figures 19 and 20 respectively. We seek a solution for the flow-field in the initial stages of the interaction, represented in the above diagrams by narrow region NZZ_1Y.

The boundary conditions for equations (3.7.2) are given, as before, by the relations on the C^- characteristic through the initial point of modification of the bore NY, and by the bore condition, (3.1.9), on the partial derivatives t_r and t_s. The former relation is derived in a manner similar to that given in §1 of this chapter and is given explicitly in §G Appendix I. Here we merely quote the result, which is

(3.7.3)

$$\text{on } s = s_n, \quad t = t_n\left(\frac{r_n + s_n}{r+s_n}\right)^{\frac{3}{2}} + (r+s_n)^{-\frac{3}{2}} \int_{r_n}^{r} \lambda(z).(z+s_n)^{\frac{1}{2}}\, dz \ ,$$

$$\text{where} \quad \lambda(r) = \frac{d}{dr}\left[X(r) + \left(\frac{3}{2}r - \frac{1}{2}s_n\right) T(r)\right] \ .$$

By differentiating the above relation for t with respect to r and multiplying throughout by a factor $\left(\frac{r+s_n}{r_n+s_n}\right)^{\frac{3}{2}}$, we obtain the following condition for U on $s = s_n$.

$$(3.7.4) \quad U(r, s_n) = -\frac{3}{2}\frac{\dot{t}_n}{(r+s_n)}\left[1 - \frac{2}{3}\frac{\lambda(r_n)}{t_n} + \frac{2}{3}\int_{r_n}^{r} \frac{\lambda'(z)}{t_n}\left(\frac{z+s_n}{r_n+s_n}\right)^{\frac{3}{2}} dz\right] .$$

The remaining boundary condition, that on the image of the bore locus, is then given in terms of the functions U and V from relations (3.7.1) and (3.1.9), and is

$$(3.7.5) \qquad \text{on } P(r, s) = 0 \quad , \qquad V(r, s) = G(r, s)\ U(r, s) .$$

The approximate method adopted consists in assuming that the required solution may be represented by an expansion in powers of $(s-s_n)$. That is, we assume that the functions U and V can be formally written as

$$(3.7.6) \qquad U(r, s) = \sum_{k=0}^{p} U_k(r).(s-s_n)^k ,$$

$$(3.7.7) \qquad V(r, s) = \sum_{k=0}^{p} V_k(r).(s-s_n)^k ,$$

where the upper limit of summation p determines the order of the approximation.

This assumption ought to give a good approximation to the flow in the early stages of decay since in this interval, the difference $s-s_n$ will be small.

The partial differential equations represented by (3.7.2) will thus be replaced by a system of interdependent first order ordinary differential equations which together with the initial conditions, determined from (3.7.4), (3.7.5), will enable the unknown functions U_i and V_i, where $i = 0, 1, 2, \ldots, p$, to be found. In fact, when $s = s_n$, we have from (3.7.6)

$$(3.7.8) \qquad U(r, s_n) = U_o(r) ,$$

and consequently $U_o(r)$ is a known function of r from relation (3.7.4). By direct substitution for $U(r, s)$, $V(r, s)$ from (3.7.6), (3.7.7) into the equations (3.7.2), we obtain the following relations.

$$(3.7.9) \quad [(r+s_n)+(s-s_n)] \sum_{k=1}^{p-1} kU_k(r).(s-s_n)^{k-1} = -\frac{3}{2} \sum_{k=0}^{p} V_k(r).(s-s_n)^k ,$$

$$(3.7.10) \quad [(r+s_n)+(s-s_n)] \sum_{k=0}^{p} V_k'(r).(s-s_n)^k = -\frac{3}{2} \sum_{k=0}^{p} U_k(r).(s-s_n)^k ,$$

where a dash denotes differentiation with respect to r. On equating coefficients of $(s-s_n)^j$, a series of ordinary differential equations in $U(r)$, $V(r)$ is obtained and are

$$(3.7.11) \qquad (r+s_n)(k+1)U_{k+1} + kU_k = -\frac{3}{2}V_k \, , \quad k \geq 0$$

$$(3.7.12) \qquad (r+s_n)V_k' + V_{k-1}' = -\frac{3}{2}U_k \, , \quad k \geq 1$$

$$(3.7.13) \qquad \text{with} \quad (r+s_n)V_o' = -\frac{3}{2}U_o \; .$$

We already know U_o. Then (3.7.13) gives V_o' from which we find V_o. We then obtain U_1 from (3.7.11) and this determines V_1' from (3.7.12). Thus the two sequences of $U(r)$ and $V(r)$ functions are constructed. At any stage U_k is determined by a purely algebraic relation but V_k has to be found as the solution of a first order ordinary differential equation. The initial condition which fixes V_k is determined from relation (3.7.5). In practice, we expand (3.7.5) about the point N which leads to the system

$$(3.7.14) \qquad V_k(r_n) = \sum_{i=0}^{k} U_i(r_n)\, G_{k-i}(r_n, s_n) \quad ,$$

where $G(r,s) = \sum_{j=0}^{\infty} G_j(r).(s-s_n)^j$, on $P(r,s) = 0$.

This form of relation (3.7.5) is meaningful so long as the function $G(r,s)$ has no singularities at the initial point of modification of the bore. $G(r,s)$ can be conveniently expressed in terms of the parameter H, defined by (1.4.15), as

$$G[r(H), s(H)] = - \left[\frac{H + \sqrt{\frac{1+H}{2}}}{H - \sqrt{\frac{1+H}{2}}} \right]^3 \quad , \; H \geq 1 \; .$$

The details of this result are given in §H Appendix I.

For the decay of a bore, the value of G at the initial point N is given by inserting the value H_n for H in the above. The initial strength of the bore is non-zero and consequently, from §4 Chapter I, $H_n > 1$. Thus the value of G at this point is always finite. However, when the formation of a bore takes place on the leading C^+ characteristic of the incident simple wave, the value of H at the initial point of formation of the bore is unity, and it is seen from the above relation that the function G has a triple pole at this point and consequently the above expansion of $G(r, s)$ around this point is not valid. This explains why the present method is limited to problems concerned with the decay of bores. It also serves to emphasize the reason why the method is applicable only to the initial stages of the process of decay and cannot be used to determine any asymptotic representation for the bore locus since in the ultimate stages of decay of the bore, $H \to 1$.

For a given value of k, U_k is obtained before V_k. This means that (3.7.14) gives $V_k(r_n)$ which is the required initial condition enabling equation (3.7.12) to be solved for V_k.

We illustrate the procedure by obtaining the first two terms for $U(r, s)$ and $V(r, s)$, that is V_o and U_1. From relations (3.7.13) and (3.7.4), we obtain immediately

$$V_o'(r) = \frac{9}{4}\frac{t_n}{(r+s_n)^2}\left[1 - \frac{2}{3}\frac{\lambda(r_n)}{t_n} + \frac{2}{3}\int_{r_n}^{r}\left(\frac{z+s_n}{r_n+s_n}\right)^{\frac{3}{2}}\frac{\lambda'(z)}{t_n}\,dz\right] .$$

After some manipulation, this equation integrates to yield the result

$$V_o(r) = V_o(r_n) + \frac{9}{4}\frac{t_n}{(r_n+s_n)}\left[\left(\frac{r-r_n}{r+s_n}\right)\left\{1 - \frac{2}{3}\frac{\lambda(r_n)}{t_n}\right\}\right]$$

$$+ t_n\frac{3}{2}(r_n+s_n)^{-2}\int_{r_n}^{r}\left(\frac{y+s_n}{r_n+s_n}\right)^{-2}\left\{\int_{r_n}^{y}\left(\frac{z+s_n}{r_n+s_n}\right)^{\frac{3}{2}}\frac{\lambda'(z)}{t_n}\,dz\right\}dy .$$

On substituting for $V_o(r_n)$ in terms of $U_o(r_n)$ from (3.7.14) with $U_o(r_n)$ as given by (3.7.4), we finally obtain,

$$\text{(3.7.15) a.}\quad V_o(r) = -\frac{3}{2}\frac{t_n}{(r_n+s_n)}\left\{1 - \frac{2}{3}\frac{\lambda(r_n)}{t_n}\right\}\left\{G_o - \frac{3}{2}\frac{r-r_n}{r+s_n}\right\}$$

$$-\frac{3}{2}(r_n+s_n)^{-2}\int_{r_n}^{r}\left(\frac{y+s_n}{r_n+s_n}\right)^{-2}\left\{\int_{r_n}^{y}\left(\frac{z+s_n}{r_n+s_n}\right)^{\frac{3}{2}}\frac{\lambda'(z)}{t_n}\,dz\right\}dy ,$$

in which $G_o = G(r_n, s_n)$.

The solution for $U_1(r)$ is then given in terms of $V_o(r)$ from (3.7.11) with $k = 0$, that is

b. $$U_1(r) = -\frac{3}{2}\,\frac{V_o(r)}{r+s_n} \quad ,$$

with $V_o(r)$ as determined by (3.7.15).

$U(r)$ and $V(r)$ functions of higher order are easily generated although the algebraic detail becomes rather more cumbersome. The expansions for $U(r,s)$ and $V(r,s)$, obtained from (3.7.6) and (3.7.7) then determine the partial derivatives of t with respect to r and s respectively at points suitably near to $s=s_n$ and consequently the independent physical variables x, t at these points may be determined as functions of the Riemann invariants which characterize the points.

In particular, to find the equation of the bore locus in the physical plane we use the differential relation

$$(3.7.16) \qquad dt = t_r dr + t_s ds = \left(\frac{r_n+s_n}{r+s}\right)^{\frac{3}{2}} [U(r,s)dr + V(r,s)ds] \ .$$

On the bore the functions $U(r,s)$, $V(r,s)$ and $G(r,s)$ are related by (3.7.5). However from equations (3.1.7)b and (3.1.9), it is seen that

$$G(r,s) = \left[\frac{\dot{\xi}(r,s) - \frac{1}{2}r + \frac{3}{2}s}{\frac{3}{2}r - \frac{1}{2}s - \dot{\xi}(r,s)}\right] \frac{dr}{ds} \quad .$$

By using the above form for $G(r,s)$ in (3.7.5) and then substituting for $V(r,s)$ in terms of $U(r,s)$ in (3.7.16) we obtain the relation

(3.7.17)
$$\frac{dt}{dr} = \frac{(r_n+s_n)^{\frac{3}{2}}\,(r+s)^{-\frac{1}{2}}\,U(r,s)}{\frac{3}{2}r - \frac{1}{2}s - \dot{\xi}(r,s)} \quad ,$$

which enables t on the bore locus to be determined as a function of the Riemann invariant r since s is defined implicitly in terms of r through the relation $P(r,s)=0$. A particular result of some interest is the initial rate of decay of the bore which is determined from the value of $\frac{dt}{dr}$ at the point N. From (3.7.17), we find

$$\left(\frac{dt}{dr}\right)_n = \frac{(r_n+s_n)\,U(r_n,s_n)}{\frac{3}{2}r_n - \frac{1}{2}s_n - \dot{\xi}(r_n,s_n)} \quad ,$$

in which $U(r_n, s_n)$ is a known function of r_n, s_n from (3.7.6) and (3.7.4). Hence we can write

(3.7.18)
$$\left(\frac{dt}{dr}\right)_n = \frac{-\frac{3}{2}t_n\left\{1 - \frac{2}{3}\frac{\lambda(r_n)}{t_n}\right\}}{\frac{3}{2}r_n - \frac{1}{2}s_n - \dot{\xi}(r_n,s_n)}$$

This result is valid for a general type of piston motion and when the incident simple wave is point-centred at the origin of the co-ordinate system, that is λ is then zero, reduces to the form (3.5.5) which was obtained by the method of the integral equation.

To obtain the solution for the initial stages of the path of the bore in the physical plane, equation (3.7.17) can either be integrated numerically, assuming that $U(r, s)$ has been found to a suitable approximation, or the bore relations (1.4.15) can be expanded around the conditions at the point N and in this manner an analytic expression for the path of the bore may be determined. We follow the latter method and to illustrate the procedure find a representation for the bore locus to terms of order, up to and including, $(s-s_n)^2$.

Using the velocity of sound c as a parameter instead of H in the bore relations (1.4.15), we may write

$$\dot{\xi} = c\sqrt{\frac{1+c^2}{2}}$$

(3.7.19)

$$\begin{matrix} r \\ s \end{matrix} = c \pm \frac{c^2-1}{2c}\sqrt{\frac{1+c^2}{2}} \quad .$$

If $q = c_n - c$, where c_n is the value of c at the point N, then the above relations when expanded in terms of q are given, after some algebra, by

$$\dot{\xi} = \dot{\xi}(r_n, s_n) - \gamma_1 q + 0(q^2) \ ,$$

(3.7.20)

$$r = r_n - (1+\alpha_1)q + \alpha_2 q^2 + 0(q^3) \ ,$$

$$s = s_n - (1-\alpha_1)q - \alpha_2 q^2 + 0(q^3) \ ,$$

where $$\gamma_1 = \frac{2c_n^2 + 1}{\sqrt{2(1+c_n^2)}} ,$$

(3.7.21) $$\alpha_1 = \frac{2c_n^4 + c_n^2 + 1}{2c_n^2 \sqrt{2(1+c_n^2)}} ,$$

and $$\alpha_2 = \frac{2c_n^6 + 3c_n^4 - 3c_n^2 - 2}{4c_n^3 (1+c_n^2) \sqrt{2(1+c_n^2)}} .$$

The above expansions are derived ab initio in §I. Appendix I. From (3.7.20) and (3.7.21), it follows that we may write

(3.7.22)

$$\left[\frac{(r_n+s_n)^{\frac{3}{2}}(r+s)^{-\frac{1}{2}}}{\frac{3}{2}r-\frac{1}{2}s-\dot{\xi}(r,s)}\right] dr = -\frac{(1+\alpha_1)(r_n+s_n)}{\frac{3}{2}r_n-\frac{1}{2}s_n-\dot{\xi}(r_n,s_n)}\left[1+\left(\frac{1}{2}c_n+\frac{\alpha_2}{1+\alpha_1}+\frac{1+2\alpha_1-\gamma_1}{\frac{3}{2}r_n-\frac{1}{2}s_n-\dot{\xi}_n}\right)q+0(q^2)\right] dq.$$

To the degree of approximation used here it is seen from (3.7.6) that $U(r,s)$ can be expressed as

(3.7.23) $$U(r,s) = U_o(r) + U_1(r)(s-s_n) + o\{(s-s_n)^2\} ,$$

in which $U_o(r)$ and $U_1(r)$ may be expanded around $r = r_n$ by Taylor's theorem to yield

(3.7.24) $$U_o(r) = U_o(r_n) + U_o(r_n)(r-r_n) + 0\{(r-r_n)^2\} ,$$

$$U_1(r) = U_1(r_n) + 0\{(r-r_n)\} .$$

From (3.7.20) it is seen that the neglected terms in (3.7.23) are of order q^2. Then using relations (3.7.4) and (3.7.15) it is possible to express $U_o(r_n)$, $U_o'(r_n)$ and $U_1(r_n)$ entirely in parameters at the point N, that is

$$U_o(r_n) = -\frac{3}{2}\,\frac{t_n}{(r_n+s_n)}\,[1-\frac{2}{3}\,\frac{\lambda(r_n)}{t_n}] \;,$$

$$(3.7.24) \qquad U_o'(r_n) = \frac{3}{2}\,\frac{t_n}{(r_n+s_n)^2}\,[1+\frac{2}{3}\{(r_n+s_n)\,\lambda'(r_n)-\lambda(r_n)\}\,\frac{1}{t_n}] \;,$$

$$U_1(r_n) = \frac{9}{4}\,\frac{t_n}{(r_n+s_n)^2}[1-\frac{2}{3}\,\frac{\lambda(r_n)}{t_n}]\;G_o(r_n) \;,$$

in which $\lambda(r_n)$, $\lambda'(r_n)$ and $G_o(r_n)$ are determined from (3.7.3) and (3.1.10).

Using relations (3.7.24) and (3.7.23) and substituting for r and s in terms of q from (3.7.20), we obtain after some algebra,

$$U(r,s) = U_o(r_n)\,[1-\{\frac{U_o'(r_n)}{U_o(r_n)}\,(1+\alpha_1)+\frac{U_1(r_n)}{U_o(r_n)}(1-\alpha_1)\}\;q+0(q^2)] \;,$$

which when multiplied by (3.7.22) then gives relation (3.7.17) as a function of the parameter q, that is

$$dt = \frac{\frac{3}{2}(1+\alpha_1)t_n\,[1 - \frac{2}{3}\,\frac{\lambda(r_n)}{t_n}]}{\frac{3}{2}r_n - \frac{1}{2}s_n - \dot{\xi}(r_n, s_n)}\,[1 + 2Aq + 0(q^2)]\,dq\ ,$$

where

$$2A = [\frac{1}{2c_n} - \frac{\alpha_2}{1+\alpha_1} + \frac{1 + 2\alpha_1 - \gamma_1}{\frac{3}{2}r_n - \frac{1}{2}s_n - \dot{\xi}(r_n, s_n)} - \frac{U_o'(r_n)}{U_o(r_n)}\,(1+\alpha_1) - \frac{U_1(r_n)}{U_o(r_n)}(1-\alpha_1)]\ .$$

The required representation for t on the bore path as a function of q is then obtained by integrating the above differential relation, thus

$$(3.7.25)\qquad t(q) - t_n = \frac{\frac{3}{2}t_n(1+\alpha_1)\,[1 - \frac{2}{3}\,\frac{\lambda(r_n)}{t_n}]}{\frac{3}{2}r_n - \frac{1}{2}s_n - \dot{\xi}(r_n, s_n)}\,[q + Aq^2 + 0(q^3)]\ .$$

This solution is valid for the initial stages of decay of a bore of any initial strength. The corresponding solution for x can now be obtained by using the relation $dx = \dot{\xi}\,dt$, in which $\dot{\xi}$ and t are known functions of q given by the relations (3.7.20) and (3.7.25) respectively. These two expansions then provide a parametric representation of the path of the bore in the physical plane.

To obtain the solution for x and t, at a particular point in the region behind the bore locus, in terms of the 'co-ordinates' of that point we proceed as follows. Each point in this region is the intersection of an r-characteristic and an s-characteristic. A general point K then lies at the intersection of two curves characterised by the values r_θ of r and s_ϕ of s , say. That is the 'point' co-ordinates of K are r_θ , s_ϕ . It is important to note that $\theta \neq \phi$ unless the point K lies on the path of the bore and we then have the relation $P(r_\theta , s_\theta) = 0$. The flow-fields in the physical and characteristic planes are as shown in Figures 19 and 20 respectively. The solution for any point $K(r_\theta , s_\phi)$ in the interior domain $YNZZ_1$ can then be derived from equation (3.7.16). There are two methods by which the solution may be determined. First by integrating along the r-characteristic on which $r = r_\theta$, that is $dr = 0$, and secondly by integrating along the s-characteristic, $s = s_\phi$ on which $ds = 0$. We choose the latter method for the following reason. Suppose that the r-characteristic $r = r_\theta$ cuts the path of the bore at the point X as shown in Figure 19, and that the s-characteristic cuts the path of the bore at the point P. Then the 'point' co-ordinates of X and P are (r_θ , s_θ) and (r_ϕ , s_ϕ) respectively. Suppose further that the solution for t on the bore, (3.7.25), is valid for $s < s_\theta$. If we integrate along the characteristic $r = r_\theta$ then we shall require the value of t at the point $X (r_\theta , s_\theta)$. But the value of t at this point will be outside of the range of the solution (3.7.25). Accordingly, we shall integrate (3.7.16) along the s-characteristic, $s = s_\phi$ and in this way ensure that the value of t on the bore locus is within the approximation (3.7.25). The limits of integration of r are $r_\theta \le r \le r_\phi$, where r_ϕ is related to s_ϕ through $P(r_\phi, s_\phi) = 0$, thus from (3.7.16),

$$\text{(3.7.26)} \qquad dt(r, s_\phi) = \left(\frac{r_n + s_n}{r + s_\phi}\right)^{\frac{3}{2}} U(r, s_\phi)\, dr \ .$$

To the order of approximation used here, we have

$$\text{(i)} \quad U(r, s_\phi) = U_o(r) + U_1(r) \,.\, (s_\phi - s_n) + O\{(s_\phi - s_n)^2\} \ ,$$

in which $U_o(r)$ and $U_1(r)$ are known functions of r given by (3.7.4) and (3.7.15) respectively.

$$\text{(ii)} \quad \left(\frac{r_n + s_n}{r + s_\phi}\right)^{\frac{3}{2}} = \left(\frac{r_n + s_n}{r + s_n}\right)^{\frac{3}{2}} \left[1 - \frac{3}{2}\left(\frac{s_\phi - s_n}{r + s_n}\right) + O\{(s_\phi - s_n)^2\}\right] \ .$$

The latter substitution is necessary as the eventual solution for $t(r_\theta, s_\phi)$ will then be significant to terms of order $(s_\phi - s_n)^2$.

On substituting the above relations into (3.7.26) and performing the integration, we obtain

$$\text{(3.7.27)} \quad t(r_\theta, s_\phi) = t(r_\phi, s_\phi) - \int_{r_\theta}^{r_\phi} \left(\frac{r_n + s_n}{r + s_n}\right)^{\frac{3}{2}} \left[U_o(r) + \left\{U_1(r) - \frac{3}{2}\frac{U_o(r)}{(r + s_n)}\right\}(s_\phi - s_n) + O\{(s_\phi - s_n)^2\}\right] dr \ ,$$

as the required solution.

For any given incident simple wave the above equation may be integrated explicitly. The solution for $t(r_\phi, s_\phi)$ is given by (3.7.25) in terms of the value of the parameter q at the point $P(r_\phi, s_\phi)$. Alternatively we may substitute for q at this point in terms of r_ϕ and s_ϕ from the relations (3.7.19). In fact, from the definition of q, we have

$$q(\phi) = \frac{1}{2}[r_n - r_\phi + s_n - s_\phi] ,$$

and relation (3.7.27) is then expressed entirely in terms of the point co-ordinates r_θ and s_ϕ and is valid up to and including terms of order $(s_\phi - s_n)^2$.

The higher order terms in the above solutions, that is (3.7.25) and (3.7.27), can be obtained by a procedure similar to that illustrated above although the algebraic details will become more cumbersome. It might be noted, however, that these effects would be minimised if the method were modified for numerical computation.

CHAPTER IV

§1. GENERAL INTRODUCTION TO CHAPTERS IV, V.

In the following chapters, the problems of the formation and decay of shock waves are considered. The mathematical formulation of these problems is as detailed in Chapter II. The flow in the region behind the modified shock locus is non-isentropic and therefore the full system of equations (1.1.8) must be used. In the present chapter, an approximate solution to the problem of the decay of a shock wave by a point-centred simple rarefaction wave is presented. This solution is based on perturbation theory. Such an approach is not new. In general, the equations of the flow-field are linearised about a region of constant state and consequently the characteristics of the perturbed system are recti-linear and parallel. Here, however, the equations are linearised about a uniform state; the incident simple wave. This procedure was developed by Gundersen [1958] who determined the solution of the first order perturbation quantities in terms of arbitrary functions of the characteristic variables. The solution has been utilised by Gundersen for the problem of the decay of a shock wave in which the piston is given a small prescribed perturbation and it is required to determine the consequent variation in the path of the shock wave. The solution for the first order perturbation quantities in terms of arbitrary functions is used here to derive a relationship on the shock locus. From this relationship, a second order ordinary differential equation for a function $\epsilon(\tau)$, assumed to be small, may be derived when the boundary conditions for the flow quantities on the shock locus are applied. The solution of this equation with the appropriate initial conditions for $\epsilon(\tau)$ then determines completely that part of the flow-field bounded by the reflection of the simple wave from the shock locus and the reflection of the backward-propagated pressure wave from the piston path. A feature of the solution is that the initial rate of change of velocity of the shock wave may be determined. This result is used in Chapter VI as a standard by which to assess the relative values of the corresponding solutions developed from certain other approximate theories.

In §7, an exact relationship is derived which involves the flow quantities in the region 'just behind' the shock wave. Essentially, this relationship relates directional derivatures along the shock locus and along the C^- characteristics and it may be written in two forms. From these forms, certain approximate relations are derived which are associated with the approximate theories of Friedrichs (1948), Pillow (1949), Whitham (1958) and Rościszewski (1960). The derivation and discussion of those approximate relations constitute the material of Chapter V.

§2. STATEMENT OF THE PROBLEM AND THE SOLUTION IN TERMS OF ARBITRARY FUNCTIONS OF THE PERTURBATION OF A POINT-CENTRED SIMPLE WAVE

If the piston is pushed with uniform velocity u_1 into the stagnant fluid, beginning from the point $x = x_a$, $t = t_a$ where $x_a = u_1 t_a$ then the equation of the consequent constant strength shock wave is

$$x - x_a = \dot{\xi}(N)\,(t - t_a)\ ,$$

where $\dot{\xi}(N)$ is the constant velocity of the shock front and is related to u_1 and the velocity of sound in the stagnation region ahead of the wave front, c_o, through the Rankine-Hugoniot relations (1.3.5).

If the piston is stopped impulsively at the origin of the co-ordinate system, a point-centred rarefaction wave is generated at O which first interacts with the shock front at the point N, where $x = x_n$ and $t = t_n$. The situation in the physical plane is then as illustrated in Figure 8.

Suffices 'sw' and '1' are used to denote flow quantities in the simple wave region ONF and in the uniform region OAN behind the constant strength shock respectively. The Riemann invariants are specified by α and β and are as defined by (1.1.9).

From the simple wave relationships as given in Chapter II, we have the following results,

$$u_{sw} + c_{sw} = \frac{x}{t}$$

$$-\frac{u_{sw}}{2} + \frac{c_{sw}}{\gamma-1} = \beta_1 = -\frac{u_1}{2} + \frac{c_1}{\gamma-1} .$$

Thus,

(4.2.1) $$u_{sw} = \frac{2}{\gamma+1}\left[\frac{x}{t} - (\gamma-1)\,\beta_1\right] ,$$

$$c_{sw} = \frac{\gamma-1}{\gamma+1}\left[\frac{x}{t} + 2\beta_1\right]$$

and $$\alpha_{sw} = \frac{2}{\gamma+1}\left[\frac{x}{t} + \frac{3-\gamma}{2}\,\beta_1\right] .$$

The problem is to determine the locus of the shock front NS after interaction with the simple wave and the flow in the region behind the modified shock front. For later convenience we note that in Figure 8, NT is the linear continuation of the undisturbed shock path AN.

The solution in the region FNS will be found on the basis of a first order perturbation of the incident simple wave. This region is partioned by the particle path NE through the initial point of modification of the shock front. Since the entropy remains constant on the particle paths it follows that the entropy variations are confined to the region RNS. We have therefore the following six regions:

(i) OAN , a region of uniform flow in which $u = u_1$ and $c = c_1$.

(ii) ONF , a point-centred simple wave domain with u and c given by (4.3.1).

(iii) OFB , a region of uniform flow in which $u = u_2$, $c = c_2$ with the Riemann invariant β_2 given by $\beta_2 = \beta_1$.

(iv) FNE , a region consisting of a simple wave plus an isentropic perturbation.

(v) FNS , a region consisting of a simple wave plus a non-isentropic perturbation.

(vi) BFG , a region of uniform state plus an isentropic perturbation.

The features of the above regions have been described in §1 Chapter I. The boundaries NE and NS are floating but on the basis of a first order theory we may replace them by their unperturbed positions for the purposes of applying boundary conditions.

We now write down the governing equations and formulate the boundary value problem. The flow in the region behind the modified shock is linearised by assuming that it can be represented as a first order perturbation of the incident simple wave. This is a reasonable approximation to make provided that we realise that the range of applicability of the resulting solution will be determined by the initial strength of the shock wave. With this assumption then, we may write

$$u = u_{sw} + u', \quad c = c_{sw} + c', \quad \alpha = \alpha_{sw} + \alpha', \quad \beta = \beta_{sw} + \beta', \quad S = S_1 + S',$$

where S_1 is the constant value of the entropy in the incident simple wave domain and a dashed symbol denotes a perturbation quantity.

On substituting for u, c, α, β and S in the equations of one-dimensional, unsteady flow (1.1.10) from the above and retaining only terms of the first order in the perturbation quantities we obtain, after some algebra, the following equations which govern the behaviour of the perturbed system.

$$\alpha'_{,t} + (u_{sw}+c_{sw})\alpha'_{,x} + \frac{1}{2}[(\gamma+1)\alpha' - (4-\gamma)\beta']\alpha_{sw,x} = \frac{c_{sw}}{2\gamma(\gamma-1)c_v} S'_{,x} \quad , \tag{4.2.2}$$

$$\beta'_{,t} + (u_{sw}-c_{sw})\beta'_{,x} + \frac{1}{2}[(3-\gamma)\alpha' - (\gamma+1)\beta']\beta_{sw,x} = \frac{-c^2_{sw}}{2\gamma(\gamma-1)c_v} S'_{,x} \quad , \tag{4.2.3}$$

$$S'_{,t} + u_{sw} S'_{,x} = 0 \quad , \tag{4.2.4}$$

where a comma denotes partial differentiation with respect to the variable written as a suffix.

The above system of equations is linear and has three distinct families of fixed characteristics, namely the α, β characteristics and the particle paths of the unperturbed incident simple wave. In particular, equation (4.2.4) expresses the fact that the entropy variations are constant on the particle paths of the unperturbed motion, given by

$$\frac{dx}{dt} = u_{sw} \quad .$$

On substituting for u_{sw} as a function of x and t from (4.2.1) and then integrating the resulting equation, it is evident that the function

$$\left(\frac{x}{t} + 2\beta_1\right) t^{\frac{\gamma-1}{\gamma+1}} = \text{constant} ,$$

on the particle paths.

Thus we may choose the particle paths of the system to be the curves $y =$ constant, where

$$(4.2.5) \qquad y = \left(\frac{\gamma-1}{\gamma+1}\right)^2 \left(\frac{x}{t} + 2\beta_1\right)^2 t^{2\frac{\gamma-1}{\gamma+1}} = c_{sw}^2 \, t^{2\frac{\gamma-1}{\gamma+1}} .$$

The solution of (4.2.4) may therefore be written in terms of an arbitrary function as

$$(4.2.6) \qquad S' = S'(y) .$$

On examining the quantity $\dfrac{c_{sw}^2 S'_{,x}}{2\gamma(\gamma-1)c_v}$ on the right hand side of (4.2.2) and (4.2.3), it is seen that it may be written as $\dfrac{c_{sw} y\, S'_{,y}(y)}{t\gamma(\gamma+1)\, c_v}$ when the substitution for $y_{,x}$ is made from (4.2.5).

If the entropy variations are then defined in terms of a new arbitrary function $w(y)$, where

$$(4.2.7) \qquad \frac{dw(y)}{dy} = \frac{y\, S'_{,y}(y)}{2\gamma(\gamma+1)\, c_v} ,$$

it is then evident that

$$(4.2.8) \qquad \frac{c_{sw}^2 S,_x}{2\gamma(\gamma-1)c_v} = \frac{2c_{sw}}{t} \frac{dw(y)}{dy} .$$

This new definition of the arbitrary function characterising the entropy variations will prove to be convenient when the equations (4.2.2) and (4.2.3) are integrated. It should be noted that (4.2.7) fixes the definition of $w(y)$ apart from a constant of integration. However, as will be shown later, this constant can be regarded as zero.

We require also another function $Z(x,t)$ which is constant on the β-characteristics of the unperturbed flow. In the same manner as given above, it may be readily verified that an appropriate form for $Z(x,t)$ is

$$(4.2.9) \qquad Z(x,t) = \left[\frac{\gamma-1}{\gamma+1}\left(\frac{x}{t} + 2\beta_1\right]^{\frac{\gamma+1}{\gamma-1}} t^2 = \left[\left(\frac{\gamma p_1}{\rho_1}\right)^{\frac{1}{\gamma-1}}\right] \rho_{sw} c_{sw} t^2 ,$$

where ρ denotes the density.

We note the relation

$$(4.2.10) \qquad \frac{c_{sw}}{y} = Z^{-\frac{\gamma-1}{\gamma+1}} .$$

The value of y on NE and Z on NF in Figure 8 will be denoted by y_n and Z_n respectively.

Since the incident simple wave is forward-facing, the Riemann invariant β_1 is constant and also, α_{sw} is given as a function of x and t by (4.2.1). On substituting for β_1, α_{sw} and S'_x as determined above, the system of equations (4.2.2 - 4.2.4) reduces to

$$(4.2.11) \qquad t\alpha'_t + x\alpha'_x + (\alpha' - \frac{3-\gamma}{\gamma+1}\beta') = 2c_{sw}\frac{dw(y)}{dy} ,$$

$$(4.2.12) \qquad t\beta'_t + (\frac{3-\gamma}{\gamma+1}x - 4\frac{\gamma-1}{\gamma+1}\beta_1 t)\beta'_x = -2c_{sw}\frac{dw(y)}{dy} .$$

The solution for the isentropic region FNE will correspond to the complementary function of the above system whilst for the non-isentropic region ENS the particular integral will have to be added. If H is an arbitrary differentiable function, the complementary function of (4.2.12) can be written as

$$(4.2.13) \qquad \beta' = 2Z^{\frac{1}{2}}\frac{dH(Z)}{dZ} .$$

The corresponding solution for α' is easily obtained from (4.2.11) and is

$$(4.2.14) \qquad \alpha' = \frac{3-\gamma}{\gamma+1}Z^{-\frac{1}{2}}H(Z) + \frac{1}{t}G(\frac{x}{t}) ,$$

where $G(\frac{x}{t})$ is an arbitrary function.

The particular integrals of (4.2.12) and (4.2.11) are

$$(4.2.15) \qquad \beta' = \frac{\gamma+1}{\gamma-1}\frac{c_{sw}}{y}w(y) = \alpha' .$$

The explicit derivation of the above solutions is given in §A Appendix II.

The solution of equations (4.2.11), (4.2.12) for the isentropic and non-isentropic regions in terms of arbitrary functions is then as follows.

(4.2.16) Region FNE.
$$\alpha' = \frac{3-\gamma}{\gamma+1}\frac{H(Z)}{Z^{\frac{1}{2}}} + \frac{1}{t}\, G\left(\frac{x}{t}\right) ,$$
$$\beta' = 2Z^{\frac{1}{2}}\frac{dH(Z)}{dZ} .$$

(4.2.17) Region ENS.
$$\alpha' = \frac{3-\gamma}{\gamma+1}\frac{K_o(Z)}{Z^{\frac{1}{2}}} + \frac{1}{t}F\left(\frac{x}{t}\right) + \frac{\gamma+1}{\gamma-1}\frac{c_{sw}}{y}\, w(y) ,$$
$$\beta' = 2Z^{\frac{1}{2}}\frac{dK_o(Z)}{dZ} + \frac{\gamma+1}{\gamma-1}\frac{c_{sw}}{y}\, w(y) ,$$

where $K_o(Z)$ and $F(\frac{x}{t})$ are arbitrary functions and are assumed for the moment to be distinct from $H(Z)$ and $G(\frac{x}{t})$ respectively.

From the above result for the non-isentropic region we deduce immediately that the entropy variations affect $c'(x,t)$ but not $u'(x,t)$.

The function $w(y)$ is determined in terms of the entropy S' from (4.2.7) and we may write

(4.2.18)
$$w(y) = W(y) + \lambda ,$$

where

$$W(y) = \int_{y_n}^{y} \frac{u\, S'_u(u)}{2\gamma(\gamma+1)c_v}\, du \quad ,$$

and λ is a constant of integration.

The mathematics of the problem will be simplified if it is possible to choose λ such that $w(y_n)$ is zero. That this can be done is shown in the following manner. We substitute for $w(y)$ from (4. 2. 18) into the relations (4. 2. 17) and obtain

$$(4.\ 2.\ 19) \qquad \alpha' = \frac{3-\gamma}{\gamma+1} \frac{K_o(Z)}{Z^{\frac{1}{2}}} + \frac{1}{t} F(\frac{x}{t}) + \frac{\gamma+1}{\gamma-1} \frac{c_{sw}}{y} \lambda + \frac{\gamma+1}{\gamma-1} \frac{c_{sw}}{y} W(y) \quad ,$$

$$\beta' = 2Z^{\frac{1}{2}} \frac{dK_o(Z)}{dZ} + \frac{\gamma+1}{\gamma-1} \frac{c_{sw}}{y} \lambda + \frac{\gamma+1}{\gamma-1} \frac{c_{sw}}{y} W(y) \quad .$$

However, from (4. 2. 10) it is seen that the quotient $\frac{c_{sw}}{y}$ is entirely a function of Z. Consequently if a new arbitrary function $K(Z)$ is defined by

$$K(Z) = K_o(Z) + \frac{\gamma+1}{3-\gamma} \cdot \frac{\gamma+1}{\gamma-1} \lambda\, Z^{\frac{3-\gamma}{2(\gamma+1)}} \quad ,$$

then relations (4. 2. 19) reduce to

$$(4.\ 2\ 20) \qquad \alpha' = \frac{3-\gamma}{\gamma+1} \frac{K(Z)}{Z^{\frac{1}{2}}} + \frac{\gamma+1}{\gamma-1} \frac{c_{sw}}{y} W(y) + \frac{1}{t} F(\frac{x}{t}) \quad ,$$

$$\beta' = 2Z^{\frac{1}{2}} \frac{dK(Z)}{dZ} + \frac{\gamma+1}{\gamma-1} \frac{c_{sw}}{y} W(y) .$$

The term involving λ in (4. 2. 19) may therefore be absorbed into an arbitrary function $K(Z)$ which is equivalent to regarding λ as zero. Consequently from (4. 2. 18) we have $w(y) = W(y)$ with $w(y_n) = 0$. The solution for the perturbation quantities α' and β' in the isentropic and non-isentropic regions is then given in terms of arbitrary functions by (4. 2. 16) and (4. 2. 20) respectively.

§3. THE BOUNDARY CONDITIONS FOR THE DECAY OF A SHOCK IN TERMS OF A PERTURBATION FUNCTION $\epsilon(\tau)$

The boundary conditions to be satisfied by the general solution quoted above when applied to the decay of a shock wave will now be formulated. If the equation of the shock wave is written as

(4. 3. 1) $$x - x_n = \dot{\xi}(N)(t-t_n) + \epsilon(t)$$

then $\epsilon(t)$ represents the deviation of the actual path of the shock wave NS from the straight line path NT. It is convenient to introduce the parameter τ as the value of t on the undisturbed path NT . Then if $\tau = \tau_n$ at the initial point of modification of the shock, we have

(4. 3. 2) $$\epsilon(\tau_n) = 0 = \dot{\epsilon}(\tau_n)$$

and

$$x = x_n + \dot{\xi}(\tau - \tau_n) \qquad \text{on NT} \quad ,$$

where $\dot{\epsilon}(\tau)$ denotes the rate of change of $\epsilon(\tau)$ with respect to τ.

In the following work we shall neglect squares and higher powers of $\epsilon(\tau)$. The approximation thus introduced limits the discussion to the two cases:

(i) If the strength of the shock is initially sufficiently weak then the resulting solution for the path of the shock will provide a good approximation for the whole history of the shock.

(ii) For shocks of greater initial strength the assumption that $\epsilon(\tau)$ is small will be valid only for the initial stages of the process of decay of the shock wave and consequently the solution is restricted to such intervals of the shock path.

We consider the values of α and β just behind the perturbed shock locus. It is clear that they may be expressed in terms of α_1, β_1 and $\dot{\epsilon}(\tau)$. In fact, to the order considered here, we have

$$\alpha_s = \alpha_1 + \left(\frac{d\alpha}{d\xi}\right)_n \dot{\epsilon}(\tau) \; ,$$

$$\beta_s = \beta_1 + \left(\frac{d\beta}{d\xi}\right)_n \dot{\epsilon}(\tau) \; ,$$

where the suffix 's' denotes conditions on the back of the shock. The quantities $(\frac{d\alpha}{d\xi})_n$ and $(\frac{d\beta}{d\xi})_n$ are known constants determined from the Rankine-Hugoniot shock relations (1.3.5) in which u_o is taken as zero. In addition α_1, β_1 are themselves related to the initial speed of the shock $\dot{\xi}(N)$ and c_o.

The values of α' and β' at the shock are therefore given by

$$\alpha'_s = (a_1 - a_{sw})_s + (\frac{d\alpha}{d\xi})_n \dot{\epsilon}(\tau) ,$$

$$\beta'_s = (\frac{d\beta}{d\xi})_n \dot{\epsilon}(\tau) ,$$

since $\beta_{sw} = \beta_1$.

From relation (4.2.1) α_{sw} is determined as a function of x and t. The value of α_{sw} taken on the straight line path NT is thus obtained by substituting for x and t in terms of τ from (4.3.2). After some manipulation, we then obtain

$$(4.3.3) \qquad (\alpha_1 - \alpha_{sw})_s = \frac{2}{\gamma+1} (u_1 + c_1 - \dot{\xi}(N)) (1 - \frac{\tau_n}{\tau}) = \lambda(\tau) , \text{ say}$$

where the substitution $x_n = (u_1 + c_1) t_n$ has been made.

If the constants $(\frac{d\alpha}{d\xi})_n$, $(\frac{d\beta}{d\xi})_n$ are denoted by T_1, T_2 respectively then the above relations for α'_s and β'_s may be written as

$$(4.3.4) \qquad \alpha'_s = \lambda(\tau) + T_1 \dot{\epsilon}(\tau) ,$$

$$\beta'_s = T_2 \dot{\epsilon}(\tau) .$$

Finally, from the Rankine-Hugoniot shock relations, the entropy variations on the shock path are given to the order considered here, by

$$(4.3.5) \qquad S'_s = T_3 \dot{\epsilon}(\tau) ,$$

where $T_3 = (\frac{dS}{d\xi})_n$ and is therefore a known constant. The constants T_1, T_2 and T_3 are given explicitly as functions of the initial velocity of the shock $\dot{\xi}(N)$ and c_o in §B Appendix II.

Relations (4.3.4) and (4.3.5) provide the boundary conditions for α', β' and S' on the shock path, which by the normal practice of linearised theory may be regarded to hold on the undisturbed shock locus NT.

The remaining boundary conditions are specified by requirements of continuity along the characteristic NF which carries the front of the disturbance reflected back from the shock. They are

$$(4.3.6) \qquad \alpha' = 0 = \beta' \quad \text{when} \quad Z = Z_n .$$

It should be noted that the conditions (4.3.4) and (4.3.6) overspecify the boundary value problem. We require only one condition on α' on a non-characteristic line of equation (4.3.11) and one condition for β' on a non-characteristic line of (4.2.12). We therefore expect to obtain a relationship on the shock locus involving α', β' and S'. It is from this relationship that the perturbation function $\epsilon(\tau)$ can be found explicitly.

§4. THE DETERMINATION OF THE FUNCTION $\epsilon(\tau)$

When the boundary condition (4.3.6) is invoked in the solution (4.2.16) for α', then on $Z = Z_n$

$$\frac{3-\gamma}{\gamma+1} \frac{H(Z_n)}{Z_n^{\frac{1}{2}}} + \left[\frac{1}{t} G\left(\frac{x}{t}\right) \right]_{Z_n = Z_n} = 0 ,$$

and consequently

(i) $G\left(\frac{x}{t}\right) \equiv 0$,

(ii) $H(Z_n) = 0$.

From the corresponding solution for β', we have also $\left[\frac{dH(Z)}{dZ}\right]_{Z_n} = 0$.

Thus throughout the isentropic region FNE α' and β' are entirely functions of Z and are related by means of the equation

(4.4.1) $$2Z \frac{d\alpha'}{dZ} + \alpha' = \frac{3-\gamma}{\gamma+1} \beta .$$

It is clear that the solutions for α' and β' in the non-isentropic region ENS must satisfy the above relationship along the curve NE , that is $y = y_n$, where $\frac{d\alpha'}{dZ}$ is now interpreted as differentiation with respect to Z along NE . Thus from (4.2.20) we have on NE the condition

(4.4.2) $$\frac{d}{dZ} \left[K(Z) + \frac{\gamma+1}{3-\gamma} Z^{\frac{1}{2}} \left\{\frac{1}{t} F\left(\frac{x}{t}\right)\right\} + \frac{(\gamma+1)^2}{(\gamma-1)(3-\gamma)} \frac{c_{sw} . Z^{\frac{1}{2}}}{y_n} w(y_n)\right]$$

$$= K'(Z) + \frac{\gamma+1}{\gamma-1} \frac{c_{sw}}{Z^{\frac{1}{2}} y_n} w(y_n) .$$

From relations (4.2.10) and (4.2.18) we have $(c_{sw})_{y_n}$ as a function of y_n and Z and $w(y_n) = 0$, respectively. Using those conditions in the above relation it is evident that (4.4.2) reduces to

$$\frac{d}{dZ}\left[Z^{\frac{1}{2}} \frac{1}{t} F\left(\frac{x}{t}\right)\right]_{y=y_n} = 0 \ ,$$

and consequently $F(\frac{x}{t}) \equiv 0$.

It has been shown that the perturbed particle velocity $u' = \alpha' - \beta'$ is unaffected by the variations in entropy, that is u' is constant on the C^- characteristics given by $Z = \text{constant}$. Hence from equations (4.2.16) and (4.2.20) with $F(\frac{x}{t}) \equiv 0$ we have the following relation between the functions $H(Z)$ and $K(Z)$ which is valid for all Z,

$$\frac{3-\gamma}{\gamma+1}\,\frac{K(Z)}{Z^{\frac{1}{2}}} - 2Z^{\frac{1}{2}}\,\frac{dK(Z)}{dZ} = \frac{3-\gamma}{\gamma+1}\,\frac{H(Z)}{Z^{\frac{1}{2}}} - 2Z^{\frac{1}{2}}\,\frac{dH(Z)}{dZ} \ ,$$

which then yields on integrating with respect to Z

$$K(Z) - H(Z) = AZ^{-\frac{3-\gamma}{2(\gamma+1)}} \ ,$$

where A is a constant of integration.

However, $H(Z_n) = 0$ and it is seen that $K(Z_n) = 0$ when the boundary condition (4.3.6) is applied for α' in equation (4.2.20) with $F(\frac{x}{t}) \equiv 0$.

The constant of integration A therefore is zero and consequently we may identify the functions $K(Z)$ and $H(Z)$.

Thus the solutions for α' and β' in the isentropic and non-isentropic regions become

(4.4.3) Region FNE
$$\alpha' = \frac{3-\gamma}{\gamma+1} \frac{K(Z)}{Z^{\frac{1}{2}}} ,$$
$$\beta' = 2Z^{\frac{1}{2}} \frac{dK(Z)}{dZ} ,$$

(4.4.4) a. Region ENS
$$\alpha' = \frac{3-\gamma}{\gamma+1} \frac{K(Z)}{Z^{\frac{1}{2}}} + \frac{\gamma+1}{\gamma-1} \frac{c_{sw}}{y} w(y) ,$$
$$\beta' = 2Z^{\frac{1}{2}} \frac{dK(Z)}{dZ} + \frac{\gamma+1}{\gamma-1} \frac{c_{sw}}{y} w(y) .$$

It may now be seen that the perturbed values of pressure and density in the isentropic and non-isentropic regions are not in general equal. The variations in entropy in the non-isentropic region contribute to the values of the pressure and density. However, on the curve $y = y_n$, the variation in entropy is zero which implies that the pressure and density are continuous across the particle path NE. Physically this means that the flow is continuous and also that the curve $y = y_n$ is not a contact discontinuity.

We now determine the perturbation function $\epsilon(\tau)$. The solution in the region ENS must satisfy the boundary conditions (4. 3. 4) on α' and β' along the unperturbed shock path NT. Moreover, on this line the entropy S' which is related to $w(y)$ by (4. 2. 18) must satisfy the condition (4. 3. 5).

The solution for α' and β', (4. 4. 4) a., is valid for all points of the non-isentropic domain of perturbation. In particular, it is valid on the boundary given by the back of the shock path on which

$$x = x_n + (\tau - \tau_n)\,\dot{\xi}(N),$$
$$t = \tau .$$

The variables Z and y when taken on the shock path may be expressed entirely as functions of τ and from (4. 4. 4)a. we then obtain

$$\alpha'_s = \frac{3-\gamma}{\gamma+1}\frac{K(Z_s)}{Z_s^{\frac{1}{2}}} + \frac{\gamma+1}{\gamma-1}\left(\frac{c_{sw}}{y}\right)_s w(y_s) \quad ,$$

(4. 4. 4) b.

$$\beta'_s = 2Z_s^{\frac{1}{2}}\frac{dK(Z_s)}{dZ_s} + \frac{\gamma+1}{\gamma-1}\left(\frac{c_{sw}}{y}\right)_s w(y_s) \quad .$$

If the function $K(Z_s)$ is now eliminated from the above relations by differentiating with respect to τ, then, after some algebra, we obtain

$$\left[\frac{d\alpha'_s}{d\tau} - \frac{\gamma+1}{\gamma-1}\left(\frac{c_{sw}}{y}\right)_s \frac{dw(y_s)}{d\tau}\right] + \frac{1}{2}\frac{d\log Z_s}{d\tau}\left[\alpha'_s - \frac{3-\gamma}{\gamma+1}\beta'_s\right] = 0 .$$

The function $w(y_s)$ is related to S_s' by equating (4.2.18) with the constant of integration $\lambda = 0$. On substituting for $w(y_s)$ in terms of S_s' in the above relationship, the following equation relating α_s', β_s' and S_s' is derived,

$$(4.4.5) \qquad \left[\frac{d\alpha_s'}{d\tau} - \frac{(c_{sw})_s}{2\gamma(\gamma-1)c_v}\frac{dS_s'}{d\tau}\right] + \frac{1}{2}\frac{d\log Z_s}{d\tau}\left[\alpha_s' - \frac{3-\gamma}{\gamma+1}\beta_s'\right] = 0 \ .$$

However, α_s', β_s' and S_s' are known in terms of the function $\lambda(\tau)$ and the $\dot{\epsilon}(\tau)$ from relations (4.3.4) and (4.3.15). Substitution of the above quantities in (4.4.5) then yields the linear equation

$$(4.4.6)\text{a.} \qquad \left[T_1 - \frac{(c_{sw})_s}{2\gamma(\gamma-1)c_v}T_2\right]\ddot{\epsilon}(\tau) + \frac{1}{2}\frac{d\log Z_s}{d\tau}\left[T_1 - \frac{3-\gamma}{\gamma+1}T_2\right]\dot{\epsilon}(\tau)$$

$$= -\dot{\lambda}(\tau) - \frac{1}{2}\frac{d\log Z_s}{d\tau}\lambda(\tau) \ ,$$

where,

$$(4.4.6)\text{b.} \qquad (c_{sw})_s = c_1 - \frac{\gamma-1}{2}\lambda(\tau)$$

and

$$\frac{1}{2}\frac{d\log Z_s}{d\tau} = \frac{1}{\tau} + \frac{\gamma+1}{2(\gamma-1)}\left[\lambda(\tau) - \frac{2}{\gamma-1}c_1\right]^{-1}\dot{\lambda}(\tau) \ .$$

Thus we have a second order linear differential equation for $\epsilon(\tau)$ with rational functions of τ as coefficients. The equation is to be solved for values of $\tau \geq \tau_n$ with the initial conditions (4.3.2), that is

$\epsilon(\tau_n) = 0 = \dot{\epsilon}(\tau_n)$. Alternatively, the equation may be regarded as a first order linear equation for $\dot{\epsilon}(\tau)$ from which $\epsilon(\tau)$ may be found by a single quadrature. The details of the solution are given in §C Appendix II, and here we merely quote the result for $\dot{\epsilon}(\tau)$

$$\dot{\epsilon}(\tau) = (\tau - \tau_n)\left(\frac{\tau_n + b_1}{\tau + b_1}\right)^{a_1}\left(\frac{\tau_n - b_2}{\tau - b_2}\right)^{a_2}\int_0^1 \left\{\frac{r_1}{1+w_1 u}\ \frac{r_2}{1+w_2 u}\ \frac{r_3}{1+w_3 u}\right\}$$

$$(1+w_1 u)^{a_1}(1+w_2 u)^{a_2}\,du\ ,$$

where a_1, a_2 ; b_1, b_2 ; r_1, r_2 ; w_1, w_2, w_3 are constants, detailed in §C Appendix II. This solution may be expressed as an infinite sum of hypergeometric functions which is convergent for values of $\tau < 2\tau_n$. An important physical quantity that can be deduced from either of equations (4.4.5), (4.4.6) is the initial deceleration of the shock wave, $\ddot{\epsilon}(\tau_n)$. At the initial point of decay of the shock wave, $\alpha' = 0 = \beta'$ and hence in (4.4.5), we obtain

$$(4.4.8) \qquad \left[\frac{d\alpha}{d\tau}\right]_n - \frac{c_1}{2\gamma(\gamma-1)c_v}\left[\frac{dS}{d\tau}\right]_n = \left[\frac{d\alpha_{sw}}{d\tau}\right]_n .$$

From the shock relations, α and S may be expressed as functions of the shock velocity $\dot{\xi}$ and (4.4.8) then determines the initial deceleration of the shock wave. The term $\left[\frac{d\alpha_{sw}}{d\tau}\right]_n$ is obtained in terms of known quantities as follows. From relations (4.2.1), (4.3.2) we have

$$(\alpha_{sw})_s = \frac{2}{\gamma+1}\left[\frac{x_n + \dot{\xi}_n(\tau - \tau_n)}{\tau} + \frac{3-\gamma}{2}\,\beta_1\right] .$$

and hence $\left[\frac{d\alpha_{sw}}{d\tau}\right]_n = -\frac{2}{(\gamma+1)t_n}\left[u_1 + c_1 - \dot{\xi}(N)\right]$, since $x_n = (u_1+c_1)t_n$.

Equation (4.4.8) may then be written in the form

$$\left[\frac{d\dot{\xi}}{d\tau}\right]_n = -\frac{2}{(\gamma+1)t_n}\;\frac{u_1 + c_1 - \dot{\xi}(N)}{\left(\frac{d\alpha}{d\xi}\right)_n - \frac{c_1}{2\gamma(\gamma-1)c_o}\left(\frac{dS}{d\xi}\right)_n} ,$$

which may be expressed entirely as a function of the parameter X , defined by

$$X = \frac{c_o}{\dot{\xi}(N)} , \qquad 0 \leq X \leq 1 ,$$

on using the shock relations. After some manipulation, we obtain

$$(4.4.9) \qquad \left(\frac{d\dot{\xi}}{d\tau}\right)_n = \frac{4}{(\gamma+1)^2}\,\frac{c_o^2}{x_a}\,\frac{(1-X^2)}{X^2}\;\frac{\left\{\sqrt{2\gamma-(\gamma-1)X^2} - \sqrt{(\gamma-1)+2X^2}\right\}^2}{1 + X^2 + 2\sqrt{\frac{(\gamma-1)+2X^2}{2\gamma-(\gamma-1)X^2}}} ,$$

where x_a is the distance through which the piston moves before being stopped impulsively at the origin, x_a is then negative. The lower bound of X corresponds to the case when the shock wave is infinitely strong and the upper bound to a vanishingly weak shock. From (4.4.9) we observe that for very strong shocks, the initial deceleration of the shock wave is proportional to the square of the initial velocity of the shock wave.

Further discussion of this solution is reserved till Chapter VI in which a comparison is made with the corresponding result from each of several approximate theories. For this purpose it is convenient to express (4.4.9) in a form involving the time t_n rather than the distance x_a. These quantities are related by,

$$u_1(u_1 + c_1 - \dot{\xi}(N))t_n = -(\dot{\xi}(N) - u_1)x_a$$

and after some manipulation, it follows that (4.4.9) in terms of t_n is

$$(4.4.10) \qquad \left(\frac{d\dot{\xi}}{d\mathcal{T}}\right)_n = -\frac{2}{\gamma+1}\,\frac{c_o}{t_n}\,[(\gamma-1)+2X^2]^{\frac{1}{2}}\left[\frac{\sqrt{2\gamma-(\gamma-1)X^2} - \sqrt{(\gamma-1)+2X^2}}{1+X^2+2\sqrt{\dfrac{(\gamma-1)+2X^2}{2\gamma-(\gamma-1)X^2}}}\right]$$

§5. THE DETERMINATION OF K(Z), w(y) AS FUNCTIONS OF $\dot{\epsilon}$.

The solution to the problem is completed by expressing the functions K(Z) and w(y) in terms of $\dot{\epsilon}$. On the shock path,

$$y_s = [c_1 - \frac{\gamma-1}{2}\,\lambda(\mathcal{T})]^2\,\mathcal{T}^{2\frac{\gamma-1}{\gamma+1}} ,$$

$$Z_s = [c_1 - \frac{\gamma-1}{2}\,\lambda(\mathcal{T})]^{\frac{\gamma+1}{\gamma-1}}\,\mathcal{T}^2 .$$

By inspection, the functions $y_s(\mathcal{T})$, $Z_s(\mathcal{T})$ are continuous and monotonic for $\mathcal{T} \geq \mathcal{T}_n$. Consequently if $\overline{y}(y_s)$, $\overline{Z}(Z_s)$ denote the inverse functions of $y_s(\mathcal{T})$, $Z_s(\mathcal{T})$ respectively, then $\overline{y}(y_s)$, $\overline{Z}(Z_s)$ are single-valued, continuous and monotonic and on the shock path, we may write

$$\bar{y}(y_s) = \tau = \bar{Z}(Z_s) .$$

From relations (4.2.18), (4.3.5),

$$w(y_s) = \frac{1}{2\gamma(\gamma-1)c_v} \int_{y_n}^{y_s} u \frac{dS'(u)}{du} du ,$$

and $S'(\tau) = T_3 \dot{\epsilon}(\tau)$.

Consequently the function $w(y)$ is determined in terms of $\dot{\epsilon}(y)$ by

(4.5.1) $$w(y) = \frac{T_3}{2\gamma(\gamma+1)c_v} \int_{y_n}^{y} u \frac{d\dot{\epsilon}(u)}{du} du .$$

The solution for $K(Z)$ is obtained from relation (4.4.4) taken on the shock path. By subtraction, we obtain

$$\frac{dK(Z_s)}{dZ_s} - \frac{1}{2} \frac{3-\gamma}{(\gamma+1)} \frac{K(Z_s)}{Z_s} = \frac{1}{2Z_s^{\frac{1}{2}}} (\beta'_s - \alpha'_s) ,$$

which, on integrating with respect to Z_s, becomes

$$K(Z_s) = \frac{1}{2} Z_s^{\frac{3-\gamma}{2(\gamma+1)}} \int_{Z_n}^{Z_s} u^{\frac{\gamma-1}{\gamma+1}} (\beta'_s - \alpha'_s) du .$$

However, from (4.3.4), β'_s and α'_s are determined in terms of $\lambda(\tau)$ and $\dot{\epsilon}(\tau)$, and

$$\beta'_s - \alpha'_s = (T_2-T_1)\ \dot{\epsilon}\,(\tau) - \lambda\,(\tau) \quad ,$$

that is, $$\beta'_s - \alpha'_s = (T_2-T_1)\ \dot{\epsilon}\,[\bar{Z}(Z_s)] - \lambda[\bar{Z}(Z_s)] \quad .$$

The solution for $K(Z)$ is then given by

$$(4.5.2) \qquad K(Z) = \frac{1}{2} Z^{\frac{3-\gamma}{2(\gamma+1)}} \int_{Z_n}^{Z} u^{\frac{\gamma-1}{\gamma+1}} \left[(T_2-T_1)\,\dot{\epsilon}\,\{\bar{Z}(u)\} - \lambda\{\bar{Z}(u)\}\right] du \,.$$

We note that the solutions (4.5.1) , (4.5.2) satisfy respectively the conditions $w(y_n) = 0$ and $K(Z_n) = 0 = \{\frac{dK(Z)}{dZ}\}_n$ as required. The flow in the isentropic and non-isentropic zones of perturbation are then determined completely from relations (4.4.3) and (4.4.4).

The description of the flow-field is concluded by determining the flow in the region BFG of Figure 8. This region is adjacent to one of constancy DFB in which $\beta_2 = \beta_1$ and $\alpha_2 = \beta_2$. The flow in the region BFG is then either uniform or a simple wave and is determined by a perturbation procedure similar to that given above except that in this case the perturbation is of the constant state in the region OFB . If α' , β' denote the perturbation of the Riemann invariants α , β as before, then $\alpha' = \alpha - \alpha_2$ and $\beta' = \beta - \beta_2$ and also the region is isentropic. The equations determining α' and β' are then, from (1.1.10), given as

$$(4.5.3) \qquad \alpha'_{,t} + (u_2 + c_2)\alpha'_{,x} = 0$$

$$\beta'_{,t} + (u_2 - c_2)\beta'_{,x} = 0 \; .$$

The characteristics of the region are therefore rectilinear and parallel. The homogeneous system (4.5.3) admits of the following general solution in terms of two arbitrary functions I and J of one argument.

$$\alpha' = I\,[x - (u_2 + c_2)t] \quad ,$$

$$\beta' = J\,[x - (u_2 - c_2)t] \quad .$$

The boundary conditions for α' , β' are given by:

(i) On the line $Z = Z_n$, which is a non-characteristic line of the above equation for α' , $\alpha' = 0$.

(ii) On the bounding C^+ characteristic FG , that is $x = (u_2 + c_2)t$, $\beta' = 2Z^{\frac{1}{2}} \frac{dK}{dZ}(Z)$.

From the former condition it is evident that the arbitrary function, I , is identically zero. The application of the latter condition requires some manipulation. On FG , $Z = c_2^{\frac{3-\gamma}{\gamma-1}} (c_2{}^2 t^2)$. Thus we have

$$J(2c_2 t) = c_2^{\frac{3-\gamma}{2(\gamma-1)}} (2c_2 t) \frac{dK}{dZ} \left[c_2^{\frac{3-\gamma}{\gamma-1}} \{ 4c_2{}^2 t^2 \} \right] .$$

The function J is therefore given by

$$(4.5.4) \quad J[x-(u_2-c_2)t] = c_2^{\frac{3-\gamma}{2(\gamma-1)}} [x-(u_2-c_2)t] \frac{dK}{dZ} \left[\frac{c_2^{\frac{3-\gamma}{\gamma-1}}}{4} \{ x-(u_2-c_2)t \}^2 \right] .$$

It should be noted by the above solution, (4.5.4), is valid only in the region bounded by FG and the reflection of FB at the piston path.

The effect of applying the boundary conditions (4. 3. 4) on the unperturbed shock locus NT can be briefly indicated in the following manner. The change of boundary will affect only the term $\lambda(\tau)$. On the unperturbed shock locus, given by (4. 3. 1), the function $\lambda(\tau)$, defined by (4. 3. 3), will be replaced by $\Lambda(\tau)$, where

(4. 5. 5) $$\Lambda(\tau) = \lambda(\tau) + \frac{2}{\gamma+1}\,\frac{\epsilon(\tau)}{\tau}\,.$$

The values of $(c_{sw})_s$ and $\frac{1}{2}\,\frac{d \log Z_s}{d\tau}$ as given by (4. 4. 6)b will be correspondingly modified.

If the boundary conditions (4. 3. 4), (4. 3. 5) are now applied with $\lambda(\tau)$ replaced by $\Lambda(\tau)$ then on neglecting squares and higher powers of $\epsilon(\tau)$ and its derivatives, we obtain the equation

(4. 5. 6) $$[T_1 - \frac{(c_{sw})_s}{2\gamma(\gamma-1)c_v}\,T_3]\,\ddot{\epsilon}(\tau) + \frac{1}{2}\,\frac{d \log Z_s}{d}\,[T_1 - \frac{3-\gamma}{\gamma+1}\,T_2]\,\dot{\epsilon}(\tau)$$

$$= -\dot{\Lambda}(\tau) - \frac{1}{2}\,\frac{d \log Z_s}{d\tau}\,\Lambda(\tau)\,,$$

corresponding to (4. 4. 6). The terms $(c_{sw})_s$, $\frac{1}{2}\,\frac{d \log Z_s}{d\tau}$ on the left hand side of this equation remain as defined by (4. 4. 6)b, but the term $\frac{1}{2}\,\frac{d \log Z_s}{d\tau}$ on the right hand side now involves $\epsilon(\tau)$ in accordance with (4. 5. 5).

Since $$\dot{\Lambda}(\tau) = \dot{\lambda}(\tau) + \frac{2}{\gamma+1}\left[\frac{\dot{\epsilon}(\tau)}{\tau} - \frac{\epsilon(\tau)}{\tau^2}\right],$$

it is apparent that

$$\dot{\Lambda}(\tau_n) = \dot{\lambda}(\tau_n),$$

and consequently the value of $\ddot{\epsilon}(\tau_n)$ will be as determined by (4.4.6). However, if (4.5.6) is now differentiated to obtain $\dddot{\epsilon}(\tau_n)$, then the terms involving $\ddot{\Lambda}(\tau_n)$ are important. From (4.5.5), we have

$$\ddot{\Lambda}(\tau_n) = \ddot{\lambda}(\tau_n) + \frac{2}{\gamma+1}\frac{\ddot{\epsilon}(\tau_n)}{\tau_n}.$$

In a similar manner, the higher derivatives of $\epsilon(\tau)$ taken at the initial point of decay of the shock wave will be modified. This process determines essentially the valid Taylor series expansion for the path of the shock wave on the basis of a first order perturbation procedure. We have adopted the cruder method for the determination of the path of the shock wave in order to illustrate simply the nature of the details involved.

§6. AN EXACT RELATION CONNECTING THE FLOW QUANTITIES IMMEDIATELY BEHIND THE SHOCK FRONT.

If the incident simple wave is not point-centred, then the equations corresponding to (4.2.11), (4.2.12) are, in general, complicated and whilst it is not impossible it is difficult to determine a solution corresponding to that presented above. One way in which this difficulty may be surpassed is to investigate the governing equations of the flow-field from a slightly more general aspect. Exact relationships may be derived for certain flow quantities at particular points of the flow-field. In this section, one such relation is presented with a view to ascertaining its suitability as the starting point for an approximate theory for the problems described in Chapter II.

Any point on the shock locus serves as a junction for the three characteristics through that point. If differentiation with respect to time along the C^+, C^-, C^o characteristics and the shock locus, $dx = \dot{\xi}(t)dt$, is denoted by the operators D_+, D_-, D_o and D_s respectively, then

$$(4.6.1)\qquad \begin{aligned} D_+ &\equiv \frac{\partial}{\partial t} + (u+c)\frac{\partial}{\partial x} \quad ; \qquad D_- \equiv \frac{\partial}{\partial t} + (u-c)\frac{\partial}{\partial x} \\ D_o &\equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} \quad ; \qquad D_s \equiv \frac{\partial}{\partial t} + \dot{\xi}\frac{\partial}{\partial x} \ . \end{aligned}$$

Figure 21 illustrates the orientation of the characteristics through a point on the shock locus, S. Some care must be taken to distinguish the effective portions of the loci. If the operators are defined in the sense of time increasing (decreasing) then the non-dotted (dotted) portions of the respective loci are the effective ones. We shall interpret the operators in the sense of increasing time. The operator D_+ interpreted thus refers to differentiation along the non-dotted portion of the C^+ characteristic across the shock. By this, we mean using the values on the C^+ characteristic obtained by solving the Cauchy problem for data given on the shock which is a non-characteristic. From (4. 6. 1) the following identities may be derived.

(4. 6. 2)
$$D_s \equiv \frac{\dot{\xi}-u+c}{2c} D_+ + \frac{u+c-\dot{\xi}}{2c} D_- ,$$
$$D_+ \equiv 2D_o - D_- .$$

Using the former relation it then follows that the rate of change of α along the shock is related to the rates of change along the C^+, C^- characteristics through the identity

(4. 6. 3)
$$D_s \alpha \equiv \frac{\dot{\xi}-u+c}{2c} D_+ \alpha + \frac{u+c-\dot{\xi}}{2c} D_- \alpha .$$

However, from the general equations governing the flow, (4. 1. 10), we have the following equations

$$D_+ \alpha = \frac{c^2}{2\gamma(\gamma-1)c_v} \frac{\partial S}{\partial x}$$
$$D_- \alpha = - \frac{c^2}{2\gamma(\gamma-1)c_v} \frac{\partial S}{\partial x} + D_- u ,$$

where in the latter relation use has been made of the defining relation, $\alpha = \beta + u$.

Relation (4. 6. 3) may then be written as

$$(4.6.4) \qquad D_s\alpha = \frac{(\dot{\xi}-u)\,c}{2\gamma(\gamma-1)c_v}\,\frac{\partial S}{\partial x} + \frac{u+c-\dot{\xi}}{2c}\,D_- u \ .$$

On the shock locus, the rate of change of entropy is

$$D_s S = \frac{\partial S}{\partial t} + \dot{\xi}\frac{\partial S}{\partial x} \ .$$

But, from the general relations (4. 1. 10),

$$O = \frac{\partial S}{\partial t} + u\frac{\partial S}{\partial x}$$

everywhere in the flow-field.

Consequently,

$$(\dot{\xi}-u)\frac{\partial S}{\partial x} = D_s S \ .$$

Thus in equation (4. 6. 4) we then obtain, on rearranging the terms slightly,

$$(4.6.5) \qquad D_s\alpha - \frac{c}{2\gamma(\gamma-1)c_v}\,D_s S = \frac{u+c-\dot{\xi}}{2c}\,D_- u \ .$$

This result is perfectly general and is valid for any problem associated with shock waves of the type described in Chapter II or indeed for any curve $dx = \dot{\xi}(t)dt$. It relates the rate of change of α and S at a point on the curve $dx = \dot{\xi}(t)dt$ in the direction of the tangent to the curve at that point to the rate of change of u in the direction of the C^- characteristic through that point. The term on the left hand side can be regarded as a function of the velocity of the shock wave, $\dot{\xi}$, by substituting for α, c and S in terms of $\dot{\xi}$ from the Rankine-Hugoniot shock relations. However, the term on the right hand side is, in general, unknown. To determine it, the particle velocity has to be a known function of the independent variables x and t on the C^- characteristics behind the shock wave. In both the formation and decay problems described in Chapter II there is one C^- characteristic on which u is a prescribed function of x and t, that is, the C^- characteristic through the initial point of modification of the shock wave which is a member of the C^- characteristics of the incident simple wave. Relation (4. 6. 5) then suffices to determine the exact value of the initial rate of change of velocity of the shock wave. Suppose, for example, that the shock wave is decayed by a point-centred simple wave, then on the bounding C^- characteristic of the simple wave the particle velocity is determined by (4. 2. 1) and consequently we may write the right hand side of (4. 6. 5) as

$$\frac{u+c-\dot{\xi}}{2c} D_- u = \frac{u+c-\dot{\xi}}{2c} D_- u_{sw} = -\frac{2}{(\gamma+1)} \frac{(u+c-\dot{\xi})}{t} .$$

From (4. 6. 5), it then follows that the initial rate of change of velocity of the shock wave is given by the expression

$$(4.6.6) \qquad \left[D_s \alpha - \frac{c}{2\gamma(\gamma-1)c_v} D_s S\right]_n = -\frac{2}{(\gamma+1)} \frac{[u_n + c_n - \dot{\xi}(N)]}{t_n} .$$

This relation is identical with that predicted by first order perturbation theory, (4. 4. 8)b. By a simple substitution, relation (4. 6. 5) may be transformed into one which involves the Riemann invariant β rather than α. From (4.6.2),

$$D_s u \equiv \frac{\dot{\xi}-u+c}{2c} D_+ u + \frac{u+c-\dot{\xi}}{2c} D_- u$$

and consequently by substituting for $\frac{u+c-\dot{\xi}}{2c} D_- u$ in (4.6.5), we obtain, after a slight rearrangement of terms, the relation

$$(4.6.7) \qquad D_s \beta - \frac{c}{2\gamma(\gamma-1)c_v} D_s S = - \frac{\dot{\xi}-u+c}{2c} D_+ u \ .$$

A more suitable form for the right hand side of (4.6.7) is obtained by using the identity (4.6.2), which yields

$$(4.6.8) \qquad D_s \beta - \frac{c}{2\gamma(\gamma-1)c_v} D_s S = - \frac{\dot{\xi}-u+c}{2c} [2D_o u - D_- u] \ .$$

In this relation, the operators D_o and D_- act along the non-dotted portions of the particle paths and the C^- characteristics behind the shock locus.

Similar remarks may be applied to (4.6.8) as were given for (4.6.5) with evident modifications. We note, however, that (4.6.8) does not enable the initial deceleration of the shock wave to be found as we do not know the particle velocity as a function of x and t along the particle path through the initial point of modification of the shock wave although we do know its value on the C^- characteristic which passes through this point. The two relationships, (4.6.5) and (4.6.8), are discussed in Chapter V when certain approximate relations are obtained from them.

It may be noted that the general procedure adopted in this section could be applied to a wider class of interaction problem.

CHAPTER V

§1. WHITHAM'S CHARACTERISTIC 'RULE'.

Before presenting two approximate relations obtained from (4.6.5), (4.6.8) it will be advantageous to give a brief outline of an approximate procedure due to G.B. Whitham, (1958).

The problems described in Chapter II may be classified by the criterion that the shock wave is modified by disturbances which are propagated from the rear of the shock front. There is, however, the complementary class of problems where the shock is modified by the region into which it is advancing. Examples of such flows are:

(i) the motion of a shock down a non-uniform tube, and

(ii) the propagation of a shock normally through a plane distribution of density, pressure, etc.

We shall consider the former problem as a means of illustrating the approximate procedure developed by Whitham for dealing with such flows. This tentative method consists in writing the equations of motion of the flow in characteristic form and then to apply the compatability condition on one set of the characteristics to the flow quantities just behind the shock front. The Rankine-Hugoniot shock relations are then used to give a first order ordinary differential equation which determines the motion of the shock wave.

We suppose that a shock wave moves along a tube of variable crossection $A(x)$ containing a gas originally at rest with constant pressure p_o and density ρ_o. For values of $x < 0$, $A(x)$ is constant and initially the shock wave is moving with constant velocity $\dot{\xi}(N)$ in this region. When the shock reaches $x = 0$, disturbances are propagated back into the region of uniform flow, as illustrated in Figure 22, and the subsequent motion of the shock is modified.

It is easily seen that the C^- characteristics in the region AOB, indicated in Figure 22, are straight and form a simple wave with $\alpha = \alpha_1 =$ constant. We shall assume that if a reflected shock wave does form then it does so outside the region of consideration and also, that the piston is sufficiently far from the shock wave so that disturbances reflected from it may be ignored.

The characteristic form of the equations governing the flow are similar to (1.1.10) except that now there is an additional term arising from the variation of crossectional area of the tube. In particular, on a C^+ characteristic it may easily be shown that the compatability condition is

$$(5.1.1) \qquad D_+\alpha - \frac{c}{2\gamma(\gamma-1)c_v} D_+S + \frac{uc}{2(u+c)} D_+ \log A = 0 .$$

Whitham now regards this relationship as holding along the shock locus. Consequently, on replacing the operator D_+ by D_s in (5.1.1), we obtain the following relation on the shock locus.

$$(5.1.2) \qquad D_s \alpha - \frac{c}{2\gamma(\gamma-1)c_v} D_s S + \frac{uc}{2(u+c)} D_s \log A = 0 .$$

On substituting for the flow quantities α, u, c and S in terms of the velocity of the shock wave, $\dot{\xi}$, from the Rankine-Hugoniot shock relations, equation (5.1.2) may then be integrated to give the shock velocity, $\dot{\xi}$, as a function of the crossectional area, A(x). For the particular cases when the strength of the shock wave is vanishingly weak, corresponding to $\dot{\xi} \to c_o$, and infinitely strong, corresponding to $\dot{\xi} \to \infty$, simple relations giving the functional dependence of $\dot{\xi}$ on A can be obtained. The interesting aspect of those results, however, is that they correspond exactly with the ones derived from the approximate solution to this problem by R. F. Chisnell, (1955, 1957). Moreover, Chisnell's results when adapted to converging cylindrical and spherical shocks compare remarkably well with the exact similarity solutions to those problems by G. Guderley, (1942). While there are satisfactory reasons which indicate why Chisnell's approximate method leads to accurate results, there seems no ascertainable reason why the simple characteristic 'rule' should be so accurate. Whitham, however, has examined his procedure in the light of the first order perturbation solution to this problem developed by W. Chester, (1954), and found that in that theory relation (5.1.2) is satisfied exactly. That this is so, may be illustrated by applying the general procedure of §6 Chapter IV to this problem. Using the identity (4.6.2) together with the compatability conditions on the C^+, C^- characteristics of the flow, we obtain after some manipulation, the relation

$$D_s \alpha = \frac{\dot{\xi}-u+c}{2c}\left[\frac{c^2}{2\gamma(\gamma-1)c_v}\frac{\partial S}{\partial x} - \frac{uc}{2}\frac{d\log A}{dx}\right] + \frac{u+c-\dot{\xi}}{2c}\left[-\frac{c^2}{2\gamma(\gamma-1)c_v}\frac{\partial S}{\partial x} - \frac{uc}{2}\frac{d\log A}{dx} + D_- u\right],$$

where, as before, use has been made of the relation $\alpha = \beta + u$.

The above relation reduces to

$$(5.1.3) \qquad D_s \alpha = \frac{c(\dot{\xi}-u)}{2\gamma(\gamma-1)c_v}\frac{\partial S}{\partial x} - \frac{uc}{2}\frac{d\log A}{dx} + \frac{u+c-\dot{\xi}}{2c} D_- u \ .$$

However, $\frac{\partial S}{\partial x}$ and $\frac{dA}{dx}$ may be written as

$$(\dot{\xi}-u)\frac{\partial S}{\partial x} = D_s S \ ; \quad \frac{d\log A}{dx} = \frac{1}{\dot{\xi}} D_s \log A$$

and consequently (5.1.3) becomes

$$(5.1.4) \qquad D_s \alpha - \frac{c}{2\gamma(\gamma-1)c_v} D_s S + \frac{uc}{2\dot{\xi}} D_s \log A = \frac{u+c-\dot{\xi}}{2c} D_- u \ .$$

Relation (5.1.4) differs from (4.6.5) by the inclusion of the term involving the effects due to the variations in the crossectional area of the tube. However, since A is a function only of x, this term may be written in an infinite number of ways as the sum of two terms, one of which involves $D_s \log A$ and the other may involve the derivative of log A along any characteristic direction or indeed any direction. For example, we may write

$$\frac{uc}{2\dot{\xi}} D_s \log A = X D_s \log A + Y D_- \log A \ ,$$

where either X or Y may be chosen at will.

One of the consequences of this effect is that relation (5.1.4) is not unique in the sense that (4.6.5) is unique. A satisfying element of Whitham's procedure is that suitable values for X and Y are chosen by the very nature of the method. For, if we put $X = \frac{uc}{2(u+c)}$, then it follows that $Y = \frac{uc}{2}\left[\frac{u+c-\dot{\xi}}{u^2-c^2}\right]$ and relation (5.1.4) may then be written as

$$(5.1.5) \qquad D_s \alpha - \frac{c}{2\gamma(\gamma-1)c_v} D_s S + \frac{uc}{2(u+c)} D_s \log A = \frac{u+c-\dot{\xi}}{2c}\left[D_- u - \frac{uc^2}{u^2-c^2} D_- \log A\right].$$

It will be noted that the left hand side of (5.1.5) is identical to that of (5.1.2). Consequently, Whitham's procedure assumes implicitly that the terms of the left hand side annul each other. However, if we now substitute for those terms their values as given by Chester's first order perturbation theory it is evident that the right hand side of (5.1.5) is indeed zero.

§2. SOME APPROXIMATE RELATIONS

Since Whitham's procedure is so simple and leads to accurate results, it is tempting to try and extend it for other problems, in particular, the problems formulated in Chapter II.

Relations (4.6.5), (4.6.8) were established for any point of the shock locus. In each expression the left hand side is essentially a known function of the velocity and acceleration of the shock wave whilst the right hand side is in general unknown. If either side were known explicitly then the other side would be determinable, but, since the determination of one side requires in fact a knowledge of the other, recourse must be made to some approximating technique. We shall choose to make certain assumptions regarding the terms on the right hand sides of the relations and consequently the shape of the shock will be determined accordingly. We wish to emphasize that the operator D acting on a quantity produces a derivative and not a differential as the notation might seem to imply. Each relation is examined in turn and we consider firstly (4. 6. 5).

The simplest assumption to make regarding the right hand side of (4. 6. 5) is to assume that the particle velocity u and the C^-, C^0 characteristics of the region behind the modified shock front are precisely as determined by the incident simple wave. Under those conditions, we then have, using identities (4. 6. 2),

$$\frac{u+c-\dot{\xi}}{2c} D_- u = \frac{u+c-\dot{\xi}}{2c} D_- u_{sw} = D_s u_{sw} ,$$

since

$$(5.2.4) \qquad D_+ u_{sw} \equiv 2D_o u_{sw} - D_- u_{sw} = 0 .$$

Moreover, if we further assume that the incident simple wave is forward-facing, then $D_s u_{sw} = D_s \alpha_{sw}$. An approximation to (4. 6. 5) is therefore given by

(5. 2. 2) $$D_s\alpha - \frac{c}{2\gamma(\gamma-1)c_v} D_s S = D_s\alpha_{sw} .$$

We choose to regard (5. 2. 2) as an empirical relation mainly for the reason that although it is produced by the above assumptions it does not necessarily imply only those assumptions. For definiteness, we shall refer to (5. 2. 2) as Form A and it may be justified, somewhat tentatively, in the following manner. We note immediately that in the absence of entropy variations Form A reduces to

(5. 2. 3) $$D_s\alpha = D_s\alpha_{sw} .$$

At the initial point of modification of the shock wave, $\alpha = \alpha_{sw} = \alpha_1$. Subject to this initial condition, (5. 2. 3) may be integrated to yield

$$\alpha = \alpha_{sw} ,$$

everywhere along the shock locus.

It will be recalled from Chapter II, that this is a sufficient condition for Friedrich's 'simple wave' approximation. If variations in entropy are ignored, then Form A reduces to a statement of Friedrich's' theory.

Secondly, we note that Form A is exact at the initial point of modification of the shock wave. This may be seen by recalling that the first order perturbation theory of Chapter IV predicts the exact value for the initial rate of change of velocity of the shock wave and this quantity is obtained from relation (4. 4. 8) which is similar to Form A.

If precisely the same considerations are applied to (4.6.8) then, on account of relation (5.2.1), we obtain

$$(5.2.4) \qquad D_s \beta - \frac{c}{2\gamma(\gamma-1)c_v} D_s S = 0 \ .$$

This relation will be termed Form B and is formally equivalent to regarding the compatability condition on the C^- characteristics of the flow-field

$$(5.2.5) \qquad D_- \beta - \frac{c}{2\gamma(\gamma-1)c_v} D_- S = 0$$

as valid along the direction of the shock locus. Since (5.2.4), (5.2.5) are both assumed to hold, it follows that

$$D\beta = \frac{c}{2\gamma(\gamma-1)c_v} DS \ ,$$

where D denotes $\frac{d}{dt}$ along any curve. Since S is constant on the particle paths it further follows that β is constant on these curves. Consequently, β is a function of the entropy variations everywhere in the region behind the modified shock locus, the functional dependence being fixed by the requirements of the Rankine-Hugoniot conditions.

The approximate method suggested by this result is not new as it is similar to that suggested by Pillow, (1948), and forms the basis of Shock-Expansion Theory which will be discussed in §5 of this chapter. We note also, that when the entropy effects are ignored then β is constant throughout the entire flow-field with its value given by the incident simple wave and consequently the entire flow-field behind the modified shock locus may be represented by an extension of the incident simple wave.

Forms A and B have been derived by tentative methods and no estimate has been given for their physical reasonableness. The author has been unable to determine the precise orders of the errors introduced by accepting them as the starting points of approximate theories for the problems formulated in Chapter II. However, in view of the fact that they do reduce to Friedrichs' theory for isentropic flows it would seem reasonable to infer that they will be accurate for weak shocks. In addition, at the initial point of modification of the shock wave, Form A is accurate irrespective of the strength of the shock wave. The approach will be adopted to enquire how closely they are associated with known approximate methods. If the association is indeed close, then this will be regarded here as sufficient justification for their application to the problems formulated in Chapter II.

§3. FORMS A AND B AND FIRST ORDER PERTURBATION THEORY

We recall from (4.4.5) that according to the assumptions of first order perturbation theory, the following relation along the shock locus is obtained for the decay of a shock wave by a point-centred simple wave.

$$\frac{d\alpha'_s}{d\tau} - \frac{(c_{sw})_s}{2\gamma(\gamma-1)c_v}\frac{dS'_s}{d\tau} + \frac{1}{2}\frac{d\log Z_s}{d\tau}\left[\alpha'_s - \frac{3-\gamma}{\gamma+1}\beta'_s\right] = 0 ,$$

where a superscript denotes a perturbation quantity. If this relation is rewritten in terms of the defining quantities then, after some manipulation, we obtain

$$\left[\frac{d\{\alpha_s-(\alpha_{sw})_s\}}{d\tau} - \frac{(c_{sw})_s}{2\gamma(\gamma-1)c_v}\frac{dS_s}{d\tau}\right] + \frac{2}{\gamma+1}\frac{d\log Z_s}{d\tau}\left[(u+c)_s - \{(u+c)_{sw}\}_s\right] = 0 .$$

In Chapter II, it was shown that the 'simple wave' approximation implies the equality of any parameter, or group of parameters, of the simple wave and of the shock wave when taken along the shock front. In the above relation, the term $[(u+c)_s - \{(u+c)_{sw}\}_s]$ if equated to zero supplies such a condition. Consequently, on the basis of the first order perturbation theory we may conclude that if the 'simple wave' approximation is regarded as the 'basic' approximation, then a higher approximation is obtained by the relation

$$\frac{d\{\alpha_s-(\alpha_{sw})_s\}}{d\tau} - \frac{(c_{sw})_s}{2\gamma(\gamma-1)c_v}\frac{dS_s}{d\tau} = 0 ,$$

which may be rewritten as

$$\text{(5.3.1)} \qquad \frac{d\alpha_s}{d\tau} - \frac{c_s}{2\gamma(\gamma-1)c_v}\frac{dS_s}{d\tau} = \frac{d\alpha_{sw}}{d\tau} ,$$

since $(c_{sw})_s$ may be replaced by c_s according to the 'simple wave' approximation.

The basic approximation is a condition on the flow quantities behind the shock wave whilst the higher approximation is a condition imposed on the derivatives of the flow quantities along the direction of the shock locus. Relation (5. 3. 1) is identical with Form A and it shows that the basic assumption necessary for Form A is that $(u+c)_s = \{(u+c)_{sw}\}_s$. The solution for the perturbed quantities α' and β' in terms of arbitrary functions as obtained by first order perturbation theory is given by (4. 4. 4 a). From this solution, the values of α' and β' on the shock locus may be derived and are given by relations (4. 4. 4 b). If those relationships are now differentiated in the direction of the shock locus, then after some algebra, it is found that

$$(5.3.2)\quad a.\quad \frac{d\{\alpha_s-(\alpha_{sw})_s\}}{d\tau} - \frac{(c_{sw})_s}{2\gamma(\gamma-1)c_v}\frac{dS_s}{d\tau}$$

$$= \frac{3-\gamma}{\gamma+1}\frac{d}{d\tau}\left[\frac{K(Z_s)}{Z_s^{\frac{1}{2}}}\right] - \left(\frac{c_{sw}}{y}\right)_s \frac{d\log Z_s}{d\tau} W(y_s)$$

and

$$b.\quad \frac{d\beta_s}{d\tau} - \frac{(c_{sw})_s}{2\gamma(\gamma-1)c_v}\frac{dS_s}{d\tau} = 2\frac{d}{d\tau}\left[Z_s^{\frac{1}{2}}\frac{dK(Z_s)}{dZ_s}\right] - \left(\frac{c_{sw}}{y}\right)_s\frac{d\log Z_s}{d\tau}W(y_s)$$

where the functions $W(y)$ and $K(Z)$ are given by (4. 5. 1) and (4. 5. 2) respectively.

Relations (5.3.2) are essentially equivalent to Forms A and B respectively when the right hand sides are zero. In relation (a) this occurs only at the initial point of modification of the shock wave since $K(Z_n) = 0 = (\frac{dK}{dZ})_n$ and $W(y_n) = 0$. At all other points of the shock locus, the right side is non-zero. The term involving $K(Z_s)$ characterises disturbances which are sent into the fluid along the C^- characteristics and that involving $W(y_s)$ characterises only part of the disturbances that are convected along the particle paths. Consequently, Form A does not take into account the disturbances along the C^- characteristics and does not consider entirely the effect of those convected along the particle paths. In relation (5.3.2) b., the right hand side involves the second derivative of the function $K(Z_s)$. This derivative does not vanish at the initial point of modification of the shock wave and consequently Form B is not satisfied exactly even at that point.

§4. FORMS A AND B AND ROSĆISZEWSKI'S METHOD

J. Rosćiszewski, (1960), derived from the general equations of motion an approximate relation which allows the calculation of certain quantities for a variety of flows which involve non-uniform shock waves. The boundary value problems so formulated are similar to those discussed in Chapter II and in §1. Data are prescribed on a non-characteristic line and the system of partial differential equations has to be solved subject to the Rankine-Hugoniot conditions on the unknown shock locus. The relation of the above paper reduces, under suitable restrictions on the flow-field, to Whitham's characteristic 'rule' and is also associated symbolically with Forms A and B.

To indicate the manner in which this relationship is derived, the following problem is chosen. We assume that the shock moves in a tube of variable crossection $A(x)$ into a stagnation region and is also modified by disturbances from the region behind the shock front. It is clear that if the modification of the shock wave is due entirely to the variations in crossectional area of the tube, then the problem reduces to that of §1. If the tube is of constant crossectional area and the shock wave is modified by a simple wave, then the problem is similar to those discussed in Chapter II.

The flow-field in the physical plane is shown in Figure 23. L represents a given non-characteristic line on which the values of the Riemann invariant α, the entropy S and the crossectional area of the tube $A(x)$ are prescribed. S represents the unknown shock locus on which the Rankine-Hugoniot conditions are to be satisfied. The equations of motion of the flow when written in characteristic form are

$$(5.4.1)\quad \text{a.}\quad D_+\alpha - \frac{c}{2\gamma(\gamma-1)c_v} D_+S + \frac{uc}{2(u+c)} D_+\log A = 0 \;,$$

$$\text{b.}\quad D_-\beta - \frac{c}{2\gamma(\gamma-1)c_v} D_-S + \frac{uc}{2(u-c)} D_-\log A = 0 \quad .$$

Rosciszewski's procedure consists in integrating the compatability condition on the C^+ characteristics along those characteristics from L to S , which yields the result

$$\alpha_s - \alpha_\ell - \int_{S_\ell}^{S_s} \frac{c}{2\gamma(\gamma-1)c_v}\, dS + \int_{\log A_\ell}^{\log A_s} \frac{uc}{2(u+c)}\, d\log A = 0 \ ,$$

where the suffices 's' and 'ℓ' denote the values of the flow quantities at the shock locus and the given curve respectively. If a similar integration is performed along a neighbouring C^+ characteristic and then the limit of the two integrals taken as the second C^+ characteristic is allowed to approach the first, we obtain, by applying the mean value theorem, the relation

$$(5.4.2) \qquad [d\alpha]_\ell^s - \left[\frac{c}{2\gamma(\gamma-1)c_v}\, dS\right]_\ell^s + \left\{\frac{dc}{2\gamma(\gamma-1)c_v}\right\}_m [S]_\ell^s + \left[\frac{uc}{2(u+c)}\, d\log A\right]_\ell^s$$

$$+ \frac{1}{2}\left\{d\left(\frac{uc}{u+c}\right)\right\}_m [\log A]_\ell^s = 0 \ ,$$

where the suffix 'm' refers to the mean value of the quantity in the interval considered.

When the variations in the flow behind the modified shock wave are small, we may write the flow quantitites in the form $A = A_\ell + \epsilon_a$, $c = c_\ell + \epsilon_\ell$, etc., where $\epsilon_a \ll A$, $\epsilon_c \ll c$, etc. The terms in which the suffix 'm' appears can then be omitted as being of the second order. Relation (5.4.2) then reduces to the form

$$(5.4.3) \qquad d\alpha_s - \frac{c_s}{2\gamma(\gamma-1)c_v}\, dS_s + \frac{1}{2}\left[\frac{uc}{u+c}\right]_s d\log A_s$$

$$= d\alpha_\ell - \frac{c_\ell}{2\gamma(\gamma-1)c_v}\, dS_2 + \frac{1}{2}\left[\frac{uc}{u+c}\right] \; d\log A_\ell.$$

We emphasize that this relationship involves differentials and not derivatives of the flow quantities. Physically it represents the manner in which small changes in the flow quantities on the given curve L are related to the consequent small changes of those quantities on the shock locus, S . The relation has been applied by Rosciszewski to a wide variety of boundary value problems and the results obtained compare favourably with those calculated by the method of characteristics using a step-by-step routine. In general, (5.4.3) cannot be integrated to obtain an analytical representation for the undetermined shock locus. However, when the shock wave is modified entirely by variations in the crossectional area of the tube, with $A(x) =$ constant, $x < 0$, then on referring to Figure 23, the curve L may be taken at any line $x =$ constant in the region $x < 0$. The quantities α_ℓ, S_ℓ and A_ℓ are constant and (5.4.3) reduces to

$$(5.4.4) \qquad d\alpha_s - \frac{c_s}{2\gamma(\gamma-1)c_v}\, dS_s + \frac{1}{2}\left[\frac{uc}{u+c}\right]_s d\log A_s = 0 \; .$$

All terms in this relationship are referred to the shock locus and consequently integration along this curve is permissable. Relation (5.4.4) is then equivalent to Whitham's characteristic 'rule'.

If the crossectional area of the tube is constant and the shock wave is modified entirely by a simple wave in the region behind the shock front then in order to apply (5.4.3) to this problem, we require to make an additional assumption concerning the flow-field. In such problems, the velocity of the piston is specified on the piston path which we may take as the given line L provided the simple wave is not point-centred. Due to the effect of the pressure waves which are propagated back into the flow from the shock front, the values of α and β will change although the difference $\alpha-\beta$ is specified. Rosciszewski assumes that the motion is such that the reflected wave does not reach the line L. Consequently, on this line, β has its value constant and given by the incident simple wave, assumed to be forward-facing. Relation (5.4.3) then reduces to

$$(5.4.5) \qquad d\alpha_s - \frac{c_s}{2\gamma(\gamma-1)c_v}\, dS_s = d\alpha_{\ell} ,$$

since S is assumed to be constant.

The left hand side of this relation may be regarded entirely as a function of the velocity of the shock wave, $\dot{\xi}$, and the right hand side as a function of the velocity of the piston, u_p. Thus (5.4.5) is of the form

$$(5.4.6) \qquad \left(\frac{d\alpha_s}{d\dot{\xi}} - \frac{c_s}{2\gamma(\gamma-1)c_v}\,\frac{dS_s}{d\dot{\xi}}\right) d\dot{\xi} = du_p ,$$

which serves to relate small changes in the velocity of the piston to the consequent changes in the velocity of the shock wave. For example, in

Figure 23 if P_o, P_1 denote neighbouring points on L and Q_o, Q_1 denote the points defined by the intersection of the C^+ characteristics through P_o, P_1 and the shock locus respectively, then the change in velocity of the piston from the point P_o to P_1 is related to the change in velocity of the shock wave from the point Q_o to Q_1 through (5.4.6).

The grouping of the flow quantities in (5.4.5) is similar to that of Form A. However, Form A is a relation involving derivatives along the shock locus which enables the relation to be integrated to give an analytical representation for the shock locus in the physical plane. Although Form A is easier to apply than (5.4.5) one must expect it to be less accurate than the latter as more drastic assumptions are required for its derivation. Nevertheless relation (5.4.5) cannot be applied when the incident simple wave is point-centred as the velocity of the piston is then discontinuous.

When the backward propagated pressure wave from the shock wave is neglected, then the values of α and β on L are as given by the incident simple wave. Following Rosciszewski, we may then integrate the compatability condition on the C^- characteristics, given by (5.4.1)b. with A(x) constant, along a C^- characteristic from the curve L to the shock locus S. If a limiting process is then performed, similar to that previously, the following relation is obtained,

$$d\beta_s - \frac{c_s}{2\gamma(\gamma-1)c_v}\, dS_s = 0 \; .$$

This relationship applies only to quantities on the shock locus and is therefore integrable and consequently is equivalent to Form B.

§5. SHOCK-EXPANSION THEORY

The central fact of the approximate method known as Shock-Expansion Theory was discovered by A. F. Pillow (1949). Whilst investigating the formation of a shock wave by a uniformly accelerated piston, Pillow found that the variation of the Riemann invariant β along the particle paths is so small compared with the variation of α that the error incurred by neglecting it is smaller than that inherent in the simple wave approximation of Friedrichs. This result can be extended to the case when the motion of the piston is non-uniform.

If the flow-field in the physical plane is as represented in Figure 24, then through any point P between the piston path and the shock locus there are three characteristics. Let R, T and W denote respectively the points on the shock locus which are the intersections of the C^o, C^- and C^+ characteristics through P and the shock locus, then Pillow has proved that the difference between the values of β at P and R is of the order of the difference between the corresponding values of α multiplied by the cube of the greatest shock strength occuring between R and W. That is, to third order terms in the shock strength, β is constant on the particle path RP. Consequently, to this order, β may be regarded a function of the entropy S since S is constant on the particle paths. However, on the shock locus β and S are related by the Rankine-Hugoniot conditions and this serves to define the functional dependence of β on S. In shock expansion theory then, β is taken to be a function of S, $\beta = \beta(S)$, given by the shock relations. It will be recalled that this is precisely the relation predicted by Form B.

On the piston path, the entropy is constant and consequently on the basis of Shock-Expansion Theory the Riemann invariant β is also constant. The approximate theory then neglects the effect of the pressure waves which are propagated back into the fluid along the C^- characteristics. It is also easily seen that the C^+ characteristics are lines of constant velocity and pressure. If this result is now substituted into the characteristic equations of the system, (1.1.12), then the compatability condition on the C^- characteristics is satisfied identically. The solution of the system is therefore determined by the compatability conditions on the C^+ and C^o characteristics. Shock-Expansion Theory replaces a three characteristic problem in two independent variables to a two characteristic problem in two independent variables. This feature of the approximate method enables the complete solution for the flow-field to be determined in closed form. The complete solution for the flow-field is not given here as it is detailed in §D Appendix II. In particular, the shock locus is found to be represented in terms of a parameter $f(u)$ by the relations [Meyer, 1959],

$$(5.5.1) \qquad \rho_o x_s = \rho_o x_m + \int_{f_o}^{f} \exp\left\{\frac{S_p - S(u)}{2\gamma\rho_v}\right\} df \ ,$$

$$t_s = t_m + \frac{1}{a_o} \int_{f_o}^{f} \frac{1}{\xi} \exp\left\{\frac{S_p - S(u)}{2\gamma c_v}\right\} df \ ,$$

where $f(u)$ is as defined in §D Appendix II.

The above solution requires to be modified if the shock wave does not start on the leading C^+ characteristic of the incident simple wave. Shock-Expansion theory has subsequently been generalised to treat a wide class of hypersonic flows and in all cases good agreement is obtained when the results are compared numerically with the existing data from physical experiments [Eggers, Savan, Syverston, 1955]. Furthermore the underlying principles of the theory have been examined by J. J. Mahony (1955), J. J. Mahony and R. R. Skeat (1955), who showed that the errors inherent in the theory are such that their effects largely cancel. The theory may then be regarded as a suitable standard by which to assess the relative merits of other "third-order" approximate theories.

CHAPTER VI

§1. INTRODUCTION.

An approximate method for weak shock waves is derived from Form A and is applied to several of the problems formulated in Chapter II. The approximation is essentially a generalisation of the 'simple wave' approximation in which terms of order the cube of the shock strength are taken into account. The procedure adopted preserves the simplificity of that of Friedrichs and is more accurate. The results obtained are compared with those from 'simple wave' theory, shock-expansion theory and also, in the case of the decay of a shock wave, with the exact solution for the initial rate of change of decay of the shock wave. The present theory predicts results which are in good agreement with those of shock-expansion theory.

§2. EXPANSION OF THE RANKINE-HUGONIOT SHOCK RELATIONS.

In §3 Chapter I, the Rankine-Hugoniot conditions are quoted in terms of the ratio of the pressure difference across the shock wave, P. For the subsequent work, it is more convenient to express them as power series expansion in the parameter δ, defined as

$$\delta = [1 + \frac{\gamma+1}{2} P]^{\frac{1}{2}} - 1 .$$

δ is then of the same order as P when $|P| < \frac{2}{\gamma+1}$. We assume that the region ahead of the shock front is a stagnation region and hence in (1. 3. 5) the quantities u_o, S_o may be taken as zero. In terms of δ, equations (1. 3. 5) are then

$$\dot{\xi} = c_o(1+\delta) ,$$

$$u = \frac{2}{\gamma+1} c_o \delta \frac{(2+\delta)}{(1+\delta)} ,$$

$$c = \frac{c_o}{1+\delta} [(1 + \frac{4\gamma}{\gamma+1} \delta + \frac{2\gamma}{\gamma+1} \delta^2) \{ 1 + 2(\frac{\gamma-1}{\gamma+1})\delta + \frac{\gamma-1}{\gamma+1} \delta^2 \}]^{\frac{1}{2}} ,$$

$$\frac{S}{c_v} = \log [\{\frac{1 + 2(\frac{\gamma-1}{\gamma+1})\delta + \frac{\gamma-1}{\gamma+1}\delta^2}{(1+\delta)^2}\}^{\gamma} \{ 1 + \frac{4\gamma}{\gamma+1} \delta + \frac{2\gamma}{\gamma+1} \delta^2 \}]$$

The above relations are singular at the following values of δ :

$$-1 \ ; \quad -1 \pm \sqrt{\frac{2}{\gamma+1}} \ ; \quad -1 \pm \sqrt{\frac{\gamma-1}{2\gamma}} .$$

The shock wave is forward-facing with $\dot{\xi} \geq c_o$. Consequently, the following Taylor series representations are valid for $\delta < |-1 + \sqrt{\frac{\gamma-1}{2\gamma}}|$,

$$u = \frac{2k-1}{k} c_o \delta [1 - \frac{\delta}{2} + \frac{\delta^2}{2} + O(\delta^3)] ,$$

$$(6.2.1) \quad c = c_o[1 + \frac{\delta}{k} - \frac{\delta^2}{2k} + \frac{\delta^3}{k} + O(\delta^4)] ,$$

$$\frac{S}{c_v} = \frac{2(2k+1)}{3k^2} [\delta^3 + O(\delta^4)] ,$$

where $2k = \frac{\gamma+1}{\gamma-1}$.

Thus we may write,

$$\alpha = \frac{2k-1}{2} c_o [1 + \frac{2}{k}\delta - \frac{\delta^2}{k} + \frac{3}{2k}\delta^3 + O(\delta^4)] ,$$

$$\beta = \frac{2k-1}{2} c_o [1 + \frac{\delta^3}{2k} + O(\delta^4)] .$$

When $\gamma = \frac{7}{5}$, the above series are convergent for $\delta < 0.622$.

§3. AN APPROXIMATE PROCEDURE FROM FORM A.

With regard to the problems formulated in Chapter II, there are a number of methods by which an analytical solution for the path of the shock wave in the physical plane may be obtained from Form A.

(i) As suggested previously, we may substitute for α, c and S on the left hand side of Form A in terms of $\dot{\xi}$ from the Rankine-Hugoniot conditions and then use the relationship $dx = \dot{\xi}\, dt$, which is valid on the shock locus, to obtain a second order, non-linear differential equation which is to be solved subject to the initial conditions: $t = t_I$, $x = x_I$ and $(\frac{dx}{dt})_I = \dot{\xi}(I)$, where the suffix 'I' is interpreted as in Chapter II.

(ii) Instead of attempting to solve the non-linear equation described above, we may simplify the problem by linearising that equation.

(iii) Rather than employ the full form of the Rankine-Hugoniot conditions, we may use series expansions of those conditions in terms of a suitable parameter and so simplify the analysis by considering those expansions only to terms of some specified order of magnitude.

The author has attempted to solve the differential equation obtained by method (i) but, owing to the extreme non-linear nature of the equation, was unable to determine the required representation for the shock locus. Of course, this equation may be used to derive a Taylor series representation for the path of the shock wave in the neighbourhood of the initial point of modification of the shock wave. When the linearised form of the above equation is considered, then the required solution may be obtained without difficulty. The analysis in this case is similar to that of Chapter IV and for that reason is not presented. Method (iii) is simplest and is the one adopted here.

To terms of order the cube of the shock strength, the expression $\frac{c_s}{2\gamma(\gamma-1)c_v} D_s S$ may be replaced by $\frac{c_o}{2\gamma(\gamma-1)c_v} D_s S$. To this order, Form A may be integrated to yield the result

$$(6.3.1) \qquad (\alpha_{sw})_s = \alpha_s - \frac{c_o}{2\gamma(\gamma-1)c_v} (S_s - S_I) ,$$

where the suffix 'I' denotes that the quantity is taken at the initial point of modification of the shock wave. In the case of the formation of a shock wave, the initial acceleration of the piston is assumed to be non-zero. The cusp of the envelope of the C^+ characteristics then lies on the leading C^+ characteristic of the simple wave and S_i is taken as zero. For the decay of a shock wave, however, S_i is of order the cube of the initial shock strength P_n.

It is interesting to note from (6.3.1) that for the former case,

$$(\alpha_{sw})_s = \alpha_s - \frac{c_o}{2\gamma(\gamma-1)c_v} S_s$$

$$\leq \alpha_s ,$$

as the contribution from the variations in entropy is positive. However, in the latter case, $\delta \leq \delta_n$ and consequently

$$(\alpha_{sw})_s = \alpha_s - \frac{c_o}{2\gamma(\gamma-1)c_v} (S_s - S_n)$$

$$\geq \alpha_s ,$$

the contribution from the entropy term now being negative. These inequalities would be expected from purely physical considerations of the respective flow-fields. They do emphasize, nevertheless, the importance of the role played by the variations in entropy in determining the solution for the path of the shock wave. In Friedrichs' 'simple wave' approximation, the corresponding relations are $(\alpha_{sw})_s = \alpha_s$ in both cases. It is to be noted that (6.3.1) is not uniformly valid for all points on the shock wave, but only for those points in the neighbourhood of the initial point of modification of the shock wave. Consequently, for the case of a decaying shock wave (6.3.1) cannot be employed to determine the asymptotic form of the solution. This is due to the fact that Form A is a first order differential relation along a curve on which conditions at two points are known, that is, at the 'initial point' and at the 'point of infinity'. In deriving (6.3.1) the former condition is invoked. As time $t \to \infty$, the shock wave tends to an acoustic wave in a stagnation region and $S_s \to 0$, $\alpha_s \to \alpha_o$, $(\alpha_{sw})_s \to \alpha_o$. Thus if Form A is now integrated subject to this set of conditions, then to the order observed here, we obtain the relation

$$(\alpha_{sw})_s = \alpha_s - \frac{c_o}{2\gamma(\gamma-1)c_v} S_s$$

$$\leq \alpha_s .$$

In the sequel, we shall be concerned entirely with the derivation of the solution for the path of the shock wave in the neighbourhood of the initial point of modification.

By using the series expansions given in (6.2.1), equation (6.3.1) may be written in the form

(6.3.2)

a. $$(\alpha_{sw})_s = A(\delta) .$$

b. where $$A(\delta) = \frac{2k-1}{2} c_o \left[1 + \frac{2}{k}\delta - \frac{1}{k}\delta^2 + \frac{5k+2}{6k^2}\delta^3 + O(\delta^4)\right] + \frac{c_o(2k-1)^2 S_i}{4(2k+1)c_v} .$$

This relation characterises the essential difference between the approximate theories derived from Form A and from the 'simple wave' assumptions of Friedrichs which lead to the relation $(\alpha_{sw})_s = \alpha_s$. However, subject to this modification, the path of the shock wave in the physical plane may be derived in a manner similar to that employed in §4 Chapter II and accordingly the details common to the development of both theories will be omitted.

The equation of the C^+ characteristics of the incident simple wave is given by (1.2.7) and therefore the quantity $(\alpha_{sw})_s$ when taken on the shock locus, where relation (6.3.2) is applicable, yields the relation

(6.3.3) $$\{x - X(A)\} = \{\frac{\gamma+1}{2}A - \frac{3-\gamma}{2}\beta_i\}\{t - T(A)\} ,$$

which corresponds to (2.4.5) when terms of order the cube of the shock strength are neglected.

The first order differential equation corresponding to (2.4.6) which determines $\bar{t} = t - t_i$ on the shock locus as a function of the parameter δ is then given by

$$[\frac{\gamma+1}{2}A - \frac{3-\gamma}{2}\beta_i - \dot{\xi}]\frac{d\bar{t}}{dA} + \frac{\gamma+1}{2}\bar{t} = -\frac{dF}{dA} ,$$

which may be written more conveniently in the form

$$\text{a.}\qquad [\tfrac{\gamma+1}{2}A - \tfrac{3-\gamma}{2}\beta_i - \dot{\xi}\,]\frac{d\bar{t}}{d\delta} + \frac{\gamma+1}{2}\frac{dA}{d\delta}\bar{t} = -\frac{dF}{d\delta}\ ,$$

(6.3.4)

$$\text{b.}\qquad \text{where } F(\delta) = [X(A) - x_i - (\tfrac{\gamma+1}{2}A - \tfrac{3-\gamma}{2}\beta_i)\,\{T(A) - t_i\}\,]\ .$$

The corresponding solution for X on the shock path may then be determined from the relation $dx = \dot{\xi}(t)dt$. The two series provide a parametric representation for the shock locus in the physical plane. In following sections, the solutions obtained from (6.3.4) will be referred as being derived by the "present method".

§4. THE FORMATION OF A SHOCK WAVE.

(a) Parametric Solution. The motion of the piston is assumed to be governed by the relation

$$x_p = X(t_p)\ ,$$

where X satisfies the conditions (1.2.10) supplemented by the condition

$$X^{(2)} > 0\ ,$$

which ensures that the shock wave is formed on the leading C^+ characteristic of the simple wave and the notation $X^{(n)}$ is used to denote the nth derivative of $X(t_p)$ taken at the origin. The simple wave is described by the equation

$$\text{(6.4.1) a.}\qquad x = X(t_p) + (u_{sw} + c_{sw})(t-t_p)$$

and throughout this region $\beta = \beta_o = \frac{c_o}{\gamma+1}$. The co-ordinates of the initial point of the envelope of the C^+ characteristics are given by (1.2.15) and are

$$\text{b.} \quad x_m = \frac{2\,c_o^{\,2}}{(\gamma+1)X^{(2)}} , \quad t_m = \frac{2\,c_o}{(\gamma+1)X^{(2)}} .$$

The flow-field in the physical plane is shown in Figure 9. For any point in the simple wave domain, the particle velocity and velocity of sound are given in terms of the function X by

$$(6.4.2) \qquad u_{sw} = \dot{X}(t_p) , \quad c_{sw} = c_o + \frac{\gamma-1}{2}\dot{X}(t_p) .$$

In order to apply Form A, we require firstly to express the parameter t_p in terms of one of the flow quantities of the simple wave. We choose to determine t_p as a function of u_{sw}. Assuming that $X(t_p)$ has continuous derivatives to the fifth order, the function may be expanded as a power series in the parameter t_p to terms of order five, that is

$$(6.4.3) \qquad X(t_p) = t_p^{\,2}\frac{X^{(2)}}{2!} + t_p^{\,3}\frac{X^{(3)}}{3!} + t_p^{\,4}\frac{X^{(4)}}{4!} + t_p^{\,5}\frac{X^{(5)}}{5!} + O(t_p^{\,6}) ,$$

where the terms involving $X(0)$ and $X^{(1)}$ are zero as the piston starts from the origin with zero velocity.

The above expansion for $X(t_p)$ is employed since we are concerned only with the initial stages of the formation of the shock wave which depend only on the initial stages of the motion of the piston. If this expression is now differentiated w.r.t t_p then on using (6.4.2), we may write

$$(6.4.4) \qquad u_{sw} = t_p X^{(2)} + t_p^2 \frac{X^{(3)}}{2!} + t_p^3 \frac{X^{(4)}}{3!} + t_p^4 \frac{X^{(5)}}{4!} + O(t_p^5) .$$

For sufficiently small values of t_p, this series may be inverted to give t_p as a function of u_{sw}. After some manipulation, we obtain

$$(6.4.5) \qquad t_p = a_1 u_{sw} + a_2 u_{sw}^2 + a_3 u_{sw}^3 + a_4 u_{sw}^4 + O(u_{sw}^5) ,$$

where $\quad a_1 = \frac{1}{X^{(2)}} ; \; a_2 = -\frac{X^{(3)}}{2\{X^{(2)}\}^3} ; \; a_3 = \frac{\{X^{(3)}\}^2}{2\{X^{(2)}\}^5} - \frac{X^{(4)}}{6\{X^{(2)}\}^4}$

and $\quad a_4 = -\frac{X^{(5)}}{24\{X^{(2)}\}^5} - \frac{5}{8}\frac{\{X^{(3)}\}^3}{\{X^{(2)}\}^7} + \frac{X^{(3)}X^{(4)}}{12\{X^{(2)}\}^6}$.

On substituting for t_p in terms of u_{sw} from (6.4.5), it is found that $X(t_p)$ may then be written as

$$(6.4.6) \qquad X(t_p) = b_1 u_{sw}^2 + b_2 u_{sw}^3 + b_3 u_{sw}^4 + O(u_{sw}^5) ,$$

where $\quad b_1 = \frac{a_1^2}{2} X^{(2)} ; \; b_2 = a_1 a_2 X^{(2)} + \frac{a_1^3}{6} X^{(3)}$

and $\quad b_3 = \frac{a_1^2 a_2}{2} X^{(3)} + a_1 a_3 X^{(2)} + \frac{a_2^2}{2} X^{(2)} + \frac{X^{(4)}}{24}$.

Consequently, after some algebra, we may write

$$(6.4.7) \qquad X(t_p) - (u_{sw} + c_{sw})t_p = -c_o a_1 u_{sw} + f_1 u_{sw}^2 + f_2 u_{sw}^3 + f_3 u_{sw}^4 + O(u_{sw}^5)$$

where $\quad f_1 = b_1 - c_o a_2 - \frac{2k}{2k-1} a_1 \; ; \; f_2 = b_2 - c_o a_3 - \frac{2k}{2k-1} a_2$

and $\quad f_3 = b_3 - c_o a_4 - \frac{2k}{2k-1} a_3 \; .$

The points x_m and t_m are given by (6.4.1)b. and hence:

(6.4.8) $$x_m - (u_{sw} + c_{sw}) t_m = \frac{2k}{2k-1} t_m u_{sw} \; .$$

On subtracting (6.4.7) from (6.4.8), we then obtain

(6.4.9) $$\{X(t_p) - (u_{sw} + c_{sw}) t_p\} - \{x_m - (u_{sw} + c_{sw}) t_m\} = f_1 u_{sw}^2 + f_2 u_{sw}^3 + f_3 u_{sw}^4 + O(u_{sw}^5) \; ,$$

since the coefficient of u_{sw} may be shown to be identically zero.

From Form A it is deduced that the value of α_{sw} when taken along the shock locus is given in terms of the parameter δ by the function $A(\delta)$, defined by equation (6.3.2)b. with S_i now taken to be zero. The value of u_{sw} taken along the shock locus is then given by

(6.4.10) $$(u_{sw})_s = (\alpha_{sw})_s - \beta_o = A(\delta) - \beta_o = \frac{2k-1}{k} c_o \, \delta [1 - \delta + \frac{5k+2}{12k} \delta^2 + O(\delta^3)] \; .$$

The function F, defined by (6.3.4)b., is, in this case,

$$F = [\{X(t_p) - (u_{sw} + c_{sw}) t_p\} - \{x_m - (u_{sw} + c_{sw}) t_m\}]_s$$

and consequently from (6.4.9), (6.4.10) we have, after some algebra, the relation

(6.4.11) $$F(\delta) = -\frac{\nu_1}{2}\delta^2[1 + \frac{2}{3}\nu_2\delta + \frac{1}{2}\nu_3\delta^2 + O(\delta^3)] ,$$

where $$-\frac{\nu_1}{2} = (\frac{2k-1}{k}c_o)^2 f_1 \quad ; \quad \frac{2}{3}\nu_2 = \frac{f_2}{f_1}(\frac{2k-1}{k}c_o) - 1$$

and $$\frac{\nu_3}{2} = \frac{13k+4}{12k} - \frac{3}{2}\frac{f_2}{f_1}(\frac{2k-1}{k}c_o) + \frac{f_3}{f_1}(\frac{2k-1}{k}c_o)^2 ,$$

assuming that the motion of the piston is such that $f_1 \neq 0$.

Equation (6.3.4)a. may now be used to determine the parametric representation for $\bar{t}$ as a function of the parameter δ subject to the initial condition $\bar{t} = 0$, $\delta = 0$.

It is easily shown from (6.3.2)b., (6.4.11) together with the relation $\dot{\xi} = c_o(1+\delta)$, that:

(6.4.12) (i) $$\frac{\gamma+1}{2}A - \frac{3-\nu}{2}\beta_o - \dot{\xi} = c_o\delta[1 - \delta + \frac{5k+2}{6k}\delta^2 + O(\delta^3)] ,$$

(ii) $$\frac{dA}{d\delta} = \frac{2k-1}{k}c_o[1-\delta+\frac{5k+2}{4k}\delta^2 + O(\delta^3)] ,$$

(iii) $$\frac{dF}{d\delta} = -\nu_1\delta[1 + \nu_2\delta + \nu_3\delta^2 + O(\delta^3)] .$$

When the above quantities are substituted into (6.3.4)a. and the terms rearranged slightly, the following differential equation is obtained.

$$\frac{d\bar{t}}{d\delta} + \frac{2}{\delta}[1 + \frac{5k+2}{12k}\delta^2 + O(\delta^3)]\bar{t} = \frac{\nu_1}{c_o}[1 + (1+\nu_2)\delta + (\nu_2+\nu_3+\frac{k-2}{6k})\delta^2 + O(\delta^3)] .$$

When the initial condition is applied, the solution of this equation, significant to the order considered here, is seen to be

(6.4.13) $$t = t_m[1 + \mu_1\delta + \mu_2\delta^2 + \mu_2\delta^3 + O(\delta^4)] ,$$

where $$\mu_1 = \frac{2k}{3(2k-1)}\frac{X^{(2)}}{c_o^2}\nu_1 \; ; \; \mu_2 = \frac{k}{2(2k-1)}\frac{X^{(2)}}{c_o^2}\nu_1(1+\nu_2)$$

and $$\mu_3 = \frac{2k}{5(2k-1)}\frac{X^{(2)}}{\delta^2}\nu_1(\nu_2 + \nu_3 - \frac{k+4}{9k}) .$$

The corresponding solution for x on the shock locus is then found from the relation $dx(\delta) = \dot{g}(\delta)\, dt(\delta)$ and is

(6.4.14) $$x = x_m[1 + \mu_1\delta + (\mu_2 + \frac{\mu_1}{2})\delta^2 + (\mu_3 + \frac{2}{3}\mu_2)\delta^3 + O(\delta^4)] .$$

Equations (6.4.13), (6.4.14) provide a parametric representation for the path of the shock in the physical plane. The solution to this problem from the 'simple wave' approximation yields precisely the relations given above except that the term involving δ^3 is absent. Our results indicate that to this order, the initial stages process of formation depend in a sensitive way on the initial values of the derivatives of $X(t_p)$ up to and including that of the fifth order.

(b) Discussion of the solution.

To illustrate the nature of the coefficients involved in (6.4.13) and to make a comparison with the corresponding solution predicted by shock-expansion theory, it is sufficient to consider the case when the motion of the piston is governed by the equation

$$X(t_p) = \frac{1}{2} X^{(2)} t_p^{\ 2} + \frac{1}{6} X^{(3)} t_p^{\ 3} ,$$

where $X^{(2)} > 0$.

The derivatives $X^{(4)}$, $X^{(5)}$ are then identically zero and the coefficients of the powers of δ in (6.4.13) may be put into a more tractable form. If π is a parameter defined by

$$\pi = \left(\frac{2k-1}{k} c_o\right) \frac{X^{(3)}}{\{X^{(2)}\}^2} ,$$

then after some algebra the coefficients μ_1, μ_2, μ_3 may be written as

(6.4.15) $$\mu_1 = \frac{2}{3}\left(\frac{2k+1}{k} - \pi\right) \ ; \ \mu_2 = -\frac{1}{4}\left(\frac{2k+1}{k} + \frac{k+2}{k}\pi - 3\pi^2\right) \ ;$$

$$\mu_3 = \left[\frac{2(2k+1)(5k+2)}{45k^2} + \frac{8k+14}{45k}\pi + \frac{3-k}{5k}\pi^2 - \pi^3\right] .$$

The quantities $\bar{\mu}_1$, $\bar{\mu}_2$, $\bar{\mu}_3$ obtained from shock-expansion theory which correspond to μ_1, μ_2, μ_3 are derived in §E Appendix II and are

(6.4.16) $$\bar{\mu}_1 = \mu_1 \ ; \ \bar{\mu}_2 = \mu_2 \ ; \ \bar{\mu}_3 = \frac{(2k+1)(3k+1)}{15k^2} + \frac{4k+13}{30k}\pi + \frac{3-k}{5k}\pi^2 - \pi^3 .$$

Our results show that the shock wave starts at the point (x_m, t_m) with an initial velocity c_o, and initial acceleration, $\ddot{\xi} = \frac{3k^2 X^{(2)}}{(2k-1)(2k+1-k\pi)}$ and an initial rate of change of acceleration

$$\dddot{\xi} = \frac{27k^4\{X^{(2)}\}^2}{4(2k-1)^2 c_o} \left[\frac{2k+1+(k+2)\pi - 3k\pi^2}{2k+1-k\pi}\right] .$$

When the acceleration of the piston is constant, then $\pi = 0$, and the above quantities when $k = 3$, have values

$$\dot{\xi} = c_o , \quad \ddot{\xi} = 0.772\, X^{(2)} , \quad \dddot{\xi} = 21.87 \frac{\{X^{(2)}\}^2}{c_o} .$$

These values correspond exactly with the ones derived from shock-expansion theory and 'simple wave' theory. This is not surprising as the coefficient of δ^3 affects only the derivatives of the shock velocity higher than the third order. Discrepancies occur in the solutions predicted by each of the three approximate theories when the fourth derivative $\xi^{(iv)}$ is calculated. In the 'simple wave' approximation $\xi^{(iv)}$ is identically zero. To investigate the variation in the values of $\xi^{(iv)}$ when obtained by the present method and shock-expansion theory, it is sufficient to examine the difference $\mu_3 - \bar{\mu}_3$. From (6.4.15), (6.4.16), it is seen that

$$\mu_3 - \bar{\mu}_3 = \frac{(2k+1)(k+1)}{45k^2} + \frac{4k-11}{90k}\,\pi .$$

For a given value of k, $\mu_3 - \bar{\mu}_3$ is a linear function of π. In general, when π is not zero, μ_3 and $\bar{\mu}_3$ may differ significantly. However, if $\pi \ll \frac{2(2k+1)(k+1)}{(4k-11)\,k}$ then μ_3 and $\bar{\mu}_3$ are of the same order of magnitude. In Table 5 the variation of μ_3, $\bar{\mu}_3$ is indicated for values of π in the interval $-0.2 < \pi < 0.3$ with $k = 3$. When the acceleration of of the piston is constant,

$$\frac{\mu_3}{\bar{\mu}_3} = \frac{2(5k+2)}{3(3k+1)} \quad .$$

Thus as k increases through the interval $(1, \infty)$, $\frac{\mu_3}{\bar{\mu}_3}$ decreases monotonically from $\frac{7}{6}$ to $\frac{10}{9}$.

Although the discrepancy in the coefficients μ_3, $\bar{\mu}_3$ is of the order of 15%, the effect on the value of t on the shock locus for a given value of $\delta < 0.62$ will be considerably reduced as each coefficient will be multiplied by the factor δ^3. Table 6 indicates this effect. The values of t, as predicted by each of the three approximate theories, are calculated for a range of values of δ. The results are shown graphically in Figure 25. It is surprising to note that even when the value of δ is as large as 0.6, the relative variation in t as obtained from the present method and shock-expansion theory is less than one per cent. The corresponding variation in the values of t when calculated from the 'simple wave' approximation and shock-expansion theory is of order -7%, illustrating the increasing importance of the terms of order δ^3. The 'simple wave' approximation is, however, in good agreement with both the present method and shock-expansion theory for values of $\delta < 0.4$. If shock-expansion is regarded as a standard, then the 'simple wave' approximation overestimates the rate of growth of the shock wave whilst the present method slightly underestimates this quantity.

§5. THE DECAY OF A SHOCK WAVE.

(a) Solution for the shock locus. The method of solution for the decay of a shock wave by a simple rarefaction wave is illustrated here for the special case when the motion of the piston is uniform at all times except at $t = 0$ where the velocity of the piston is discontinuous. In particular, we assume that the piston is accelerated instantaneously from rest, at the point $x = x_a$, to a constant velocity u_1, thus producing a shock wave of constant strength P_1 which advances into the gas at rest with constant speed $\dot{\xi}(N)$. After a finite time $t = t_a$, the piston is suddenly stopped and the resulting simple wave is point-centred at the origin, taken for convenience at the point where the motion of the piston ceases. The subsequent behaviour of the flow-field is as detailed in §1 Chapter II and is depicted in Figure 8. This particular problem has been selected for the two following reasons:

(i) the results of Chapter IV may be used as a convenient 'standard' by which to assess both the present approximate theory and shock-expansion theory in addition to the 'simple wave' theory.

(ii) even when the incident simple wave is not point-centred no essentially new features arise in the analysis. The procedure is similar to that employed in the sequel and, as in the preceeding section, it is assumed that the motion of the piston may be represented by a Taylor series expansion in the parameter t_p. There is, however, a certain amount of algebraic detail which the author feels would tend to obscure the simplicity of the method of solution from Form A.

The equation of the C^+ characteristics of the simple wave is

$$x = (\frac{\gamma+1}{2}\alpha_{sw} - \frac{3-\gamma}{2}\beta_1)t \quad ,$$

where $\quad \beta_1 = -\frac{u_1}{2} + \frac{c_1}{\gamma-1} \quad .$

Consequently, the function F defined by (6.3.4)b. is given by the simple relation

$$(6.5.1) \qquad F(\delta) = -[x_n - \{\frac{\gamma+1}{2}A(\delta) - \frac{3-\gamma}{2}\beta_1\}t_n]$$

where $\quad A(\delta) = \frac{2k-1}{2}c_o[1 + \frac{2}{k}\delta - \frac{1}{k}\delta^2 + \frac{5k+2}{6k^2}\delta^3 + O(\delta^4)]$

$$+ c_o\frac{(2k-1)^2}{6k^2}[\delta_n^{\ 3} + O(\delta_n^{\ 4})] \quad .$$

On substituting for $\frac{dF}{d\delta}$ from (6.5.1) into (6.3.4)a., the differential equation for time $\bar{t}$ on the shock locus as a function of the parameter δ is found to be

$$[\frac{\gamma+1}{2}A - \frac{3-\gamma}{2}\beta_1 - \dot{\xi}]\frac{d\bar{t}}{d\delta} + \frac{\gamma+1}{2}\frac{dA}{d\delta}\bar{t} = -\frac{\gamma+1}{2}t\frac{dA}{d\delta} \quad ,$$

which reduces to the homogeneous equation in t

$$(6.5.2) \qquad [\frac{\gamma+1}{2}A - \frac{3-\gamma}{2}\beta_1 - \dot{\xi}]\frac{dt}{d\delta} + \frac{\gamma+1}{2}\frac{dA}{d\delta}t = 0 \quad ,$$

with initial condition, $t = t_n, \quad \delta = \delta_n$.

From (6.5.2) the quantities $(\frac{dt}{d\delta})_n$, $(\frac{d^2t}{d\delta^2})_n$ can be quickly calculated and the derivatives $(\frac{d^2x}{dt^2})_n$, $(\frac{d^3x}{dt^3})_n$ determined to the significant order in δ_n.

Since $A(\delta_n) = \alpha_1$, we have immediately,

$$(6.5.3)\ \text{a.}\quad [u_1 + c_1 - \dot{\xi}(N)]\,(\frac{dt}{d\delta})_n + \frac{\gamma+1}{2}(\frac{dA}{d\delta})_n t_n = 0 \ .$$

On differentiating (6.5.2) w.r.t δ, we then find that $(\frac{d^2t}{d\delta^2})_n$ is determined by

$$\text{b.}\quad [u_1 + c_1 - \dot{\xi}_n]\,(\frac{d^2t}{d\delta^2})_n + [(\gamma+1)\,(\frac{dA}{d\delta})_n - (\frac{d\dot{\xi}}{d\delta})_n]\,(\frac{dt}{d\delta})_n + \frac{\gamma+1}{2}(\frac{d^2A}{d\delta^2})_n t_n = 0 \ .$$

From relations (6.2.1) and (6.5.1), we obtain the following expansions in the parameter δ_n.

$$(u_1 + c_1 - \dot{\xi}_n)^{-1} = \frac{1}{c_o\delta_n}\,[1 + \delta_n - \frac{\delta_n^{\ 2}}{2k} + O(\delta_n^{\ 3})] \ ,$$

$$\frac{\gamma+1}{2}(\frac{dA}{d\delta})_n = 2c_o\,[1 - \delta_n + \frac{5k+2}{4k}\delta_n^{\ 2} + O(\delta_n^{\ 3})] \ ,$$

$$\frac{\gamma+1}{2}(\frac{d^2A}{d\delta^2})_n = -\,2\,c_o\,[1 - \frac{5k+2}{2k}\delta_n + O(\delta_n^{\ 2})] \ ,$$

$$(\gamma+1)(\frac{dA}{d\delta})_n - (\frac{d\dot{\xi}}{d\delta})_n = 3\,c_o\,[1 - \frac{4}{3}\delta_n + \frac{5k+2}{3k}\delta_n^{\ 2} + O(\delta_n^{\ 3})] \ .$$

Thus from (6.5.3)a,b we have, after some manipulation, the following representations for $(\frac{dt}{d\delta})_n$, $(\frac{d^2t}{d\delta^2})_n$ in terms of δ_n , significant to the order indicated.

$$(6.5.4)\text{a.}\qquad \left(\frac{dt}{d\delta}\right)_n = -\frac{2t_n}{\delta_n}\left[1+\frac{1}{4}\delta_n^{\,2}+O(\delta_n^{\,3})\right] ,$$

$$\text{b.}\qquad \left(\frac{d^2t}{d\delta^2}\right)_n = \frac{6t_n}{\delta_n^{\,2}}\left[1+\frac{k-2}{12k}\delta_n^{\,2}+O(\delta_n^{\,3})\right] .$$

On the shock locus, $dx = c_o(1+\delta)dt$ and consequently in terms of the above quantities the derivatives $\left(\frac{d^2x}{dt^2}\right)_n$, $\left(\frac{d^3x}{dt^3}\right)_n$ are given by

$$\left(\frac{d^2x}{dt^2}\right)_n = c_o\left[\left(\frac{dt}{d\delta}\right)_n\right]^{-1} ; \quad \left(\frac{d^3x}{dt^3}\right)_n = -c_o\left(\frac{d^2t}{d\delta^2}\right)_n\left[\left(\frac{dt}{d\delta}\right)_n\right]^{-3} ,$$

that is,

$$(6.5.5)\qquad \left(\frac{d^2x}{dt^2}\right)_n = -\frac{c_o\delta_n}{2t_n}\left[1-\frac{1}{4}\delta_n^{\,2}+O(\delta_n^{\,3})\right]$$

and

$$\left(\frac{d^3x}{dt^3}\right)_n = \frac{3c_o\delta_n}{4t_n^{\,2}}\left[1-\frac{4k+1}{6k}\delta_n^{\,2}+O(\delta_n^{\,3})\right] .$$

In the neighbourhood of the initial point of decay of the shock wave, the shock locus in the physical plane is thus given by

$$x-x_n = c_o t_n\left[(1+\delta_n)T-\frac{\delta_n}{4}\left\{1-\frac{1}{4}\delta_n^{\,2}+O(\delta_n^{\,3})\right\}T^2+\frac{\delta_n}{8}\left\{1-\frac{4k+1}{6k}\delta_n^{\,2}+O(\delta_n^{\,3})\right\}T^3 + O(T^4)\right] ,$$

where $T = \dfrac{t-t_n}{t_n}$.

The corresponding solution for $\left(\frac{d^2x}{dt^2}\right)_n$, $\left(\frac{d^3x}{dt^3}\right)_n$ from the 'simple wave' approximation may be determined as above except that the function $A(\delta)$ will be replaced by $\alpha_s(\delta)$ up to and including terms of order two in δ . It is then seen that those quantities, to the significant order in δ , are given by

(6. 5. 6) $$\left(\frac{d^2x}{dt^2}\right)_n = -\frac{c_o\delta_n}{2t_n}\,[1 + O(\delta_n^{\,2})]\,, \quad \left(\frac{d^3x}{dt^3}\right)_n = \frac{3c_o\delta_n}{4t_n^{\,2}}\,[1 + O(\delta_n^{\,2})] \ .$$

For the purpose of comparison of results, the above initial derivatives have been calculated on the basis of Shock-Expansion theory, the details being given in §F Appendix II. We quote here the final forms,

(6. 5. 7)
$$\left(\frac{d^2x}{dt^2}\right)_n = -\frac{c_o\delta_n}{2t_n}\,[1 - \frac{k-1}{2k}\,\delta_n^{\,2} + O(\delta_n^{\,3})] \ ,$$
$$\left(\frac{d^3x}{dt^2}\right)_n = -\frac{3c_o\delta_n}{4t_n^{\,2}}\,[1 - \frac{k-1}{k}\,\delta_n^{\,2} + O(\delta_n^{\,3})] \ .$$

(b) Discussion of results. The initial deceleration of the shock wave as predicted by the present theory, shock-expansion theory and the 'simple wave' theory is compared with the exact solution as determined by (4. 4. 10). As is noted in Chapter V, Form A is exact at the initial point of decay of the shock wave. However, in deriving an approximate procedure from Form A only those terms of order up to and including the cube of the shock strength are taken into consideration. Consequently, the difference in the values of $\left(\frac{d^2x}{dt^2}\right)_n$ as given by (4.4.10) and (6.5.4) is due to the effect of higher order terms in the shock strength. In Table 7 some numerical values of the initial deceleration of the shock wave as predicted by each of the three approximate theories together with the exact solution are shown for a range of values of $\delta_n \leq 0.7$, with the adiabatic index, $\gamma = 1.4$. Figure 26 illustrates the graph of those results. On the 'simple wave' approximation, the initial deceleration of the shock wave is a linear function of δ_n and we observe that this theory overestimates the rate at which the velocity of the shock wave diminishes at the initial point of decay. It is surprising, however, to note that shock-expansion theory underestimates the initial deceleration of the shock wave by a magnitude of the same order as that by which the 'simple wave' approximation overestimates it. The solution as determined by the present method is, as expected, in closer agreement with the exact solution than either of the other two approximate theories. However, even when δ_n has a value as large as 0.7, corresponding to a pressure difference of 2.675, the solutions from the approximate theories differ from the exact solution by amounts which are less than 10%; more precisely, with the 'simple wave' approximation, the discrepancy is of order 9%, with shock-expansion theory, - 8%, and with the present theory, - 4%.

In Table 8 some numerical values are shown for the initial rate of change of deceleration of the shock wave for the range of values of δ_n, $0 \leq \delta_n \leq 0.7$. Figure 27 illustrates the graph of these results. We note that the initial rate of change of deceleration of the shock wave when calculated from the present theory is slightly less than that predicted by shock-expansion theory. When $\delta_n = 0.6$, this difference amounts to 3%. However, when calculated from the 'simple wave' approximation, the initial rate of change of deceleration of the shock wave is a linear function of δ_n, and is, for $\delta_n > 0.3$, significantly greater than the values predicted by either of the third order theories. For example, for $\delta_n = 0.3$, the difference in the calculated values is of order 6% whilst for $\delta_n = 0.6$, this quantity increases to 35%, which indicates the importance of the third order terms in the shock strength.

From Tables 7 and 8, one may infer that the 'simple wave' approximation for the initial deceleration of the shock wave is relatively accurate over the observed range of values of δ_n. However, this theory considerably overestimates the rate at which the shock wave weakens for values of $\delta_n > 0.3$. The present theory shows good agreement with the exact value of the initial deceleration of the shock wave in the indicated range of values of δ_n. In fact, in this respect, it is more accurate than shock-expansion theory. With regard to the initial rate of change of deceleration of the shock wave, both theories predict similar numerical results. The analysis suggests that although the present theory may be regarded as a simple extension of Friedrichs' theory. It is nevertheless comparable with shock-expansion theory. However, even though the present theory has the advantage that it predicts the correct expansion (to third order terms) for the initial deceleration of the shock wave, it possesses the disadvantage that it is not uniformly valid over the whole history of the shock wave.

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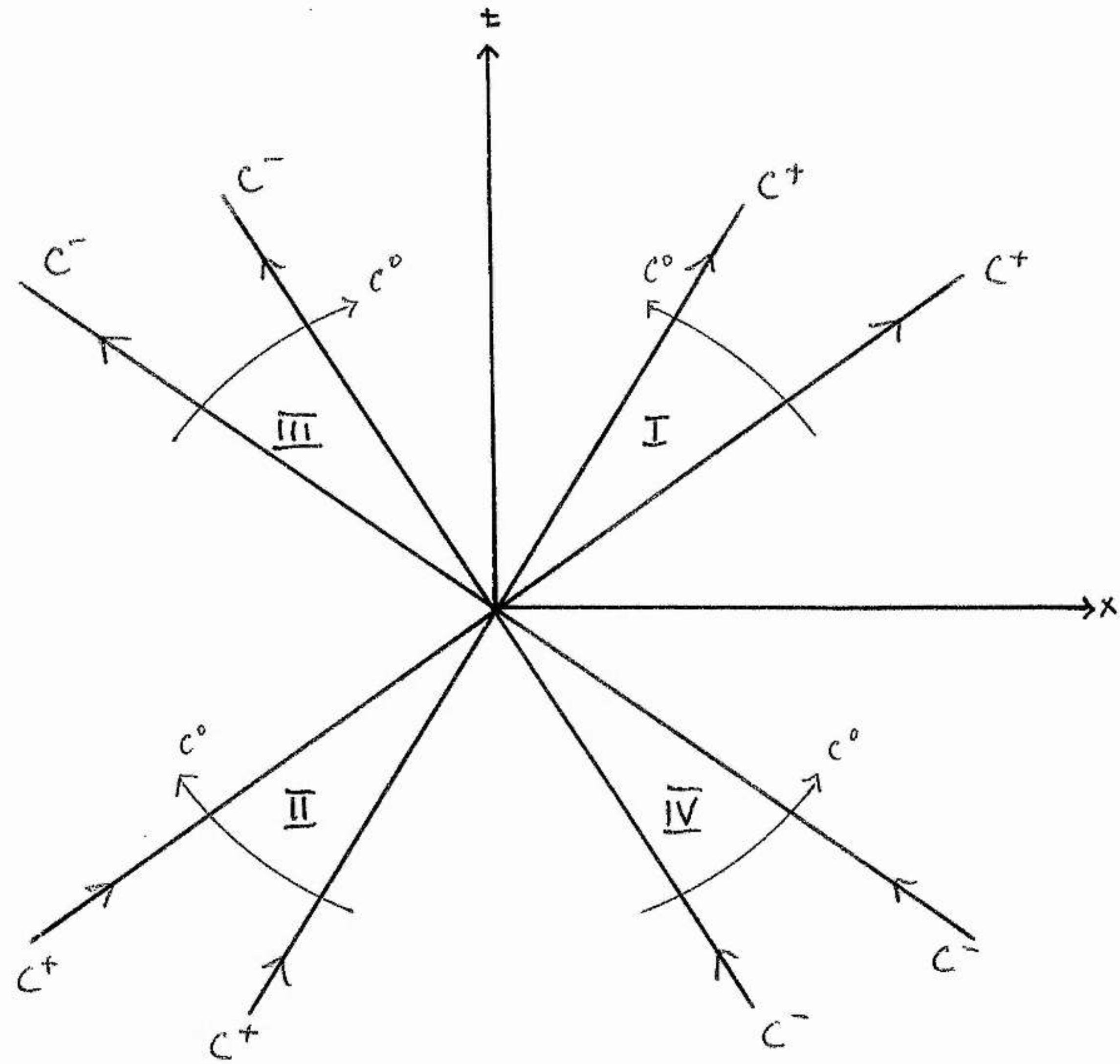


I. Forward Rarefaction Wave

II. Forward Compression Wave

III. Backward Rarefaction Wave

IV. Backward Compression Wave

Figure 1

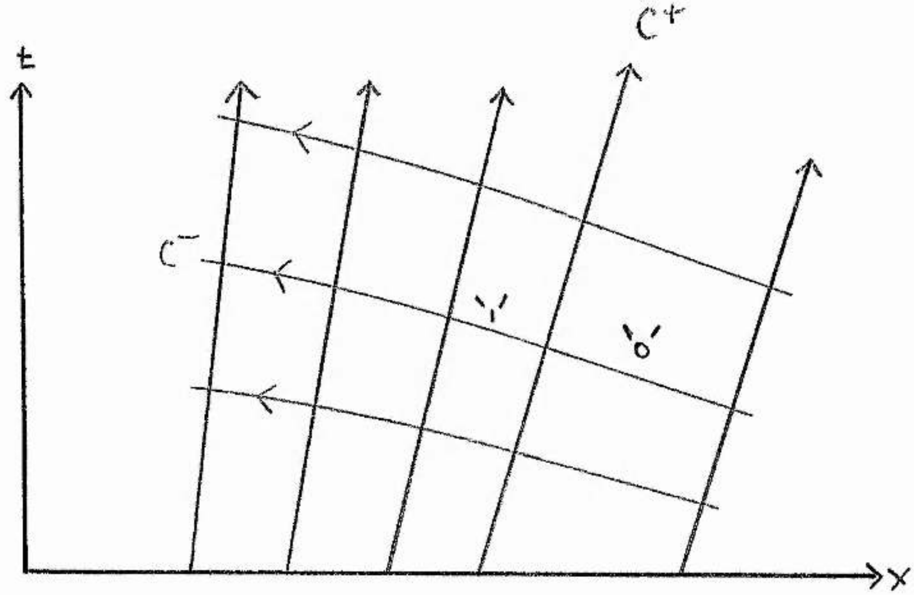
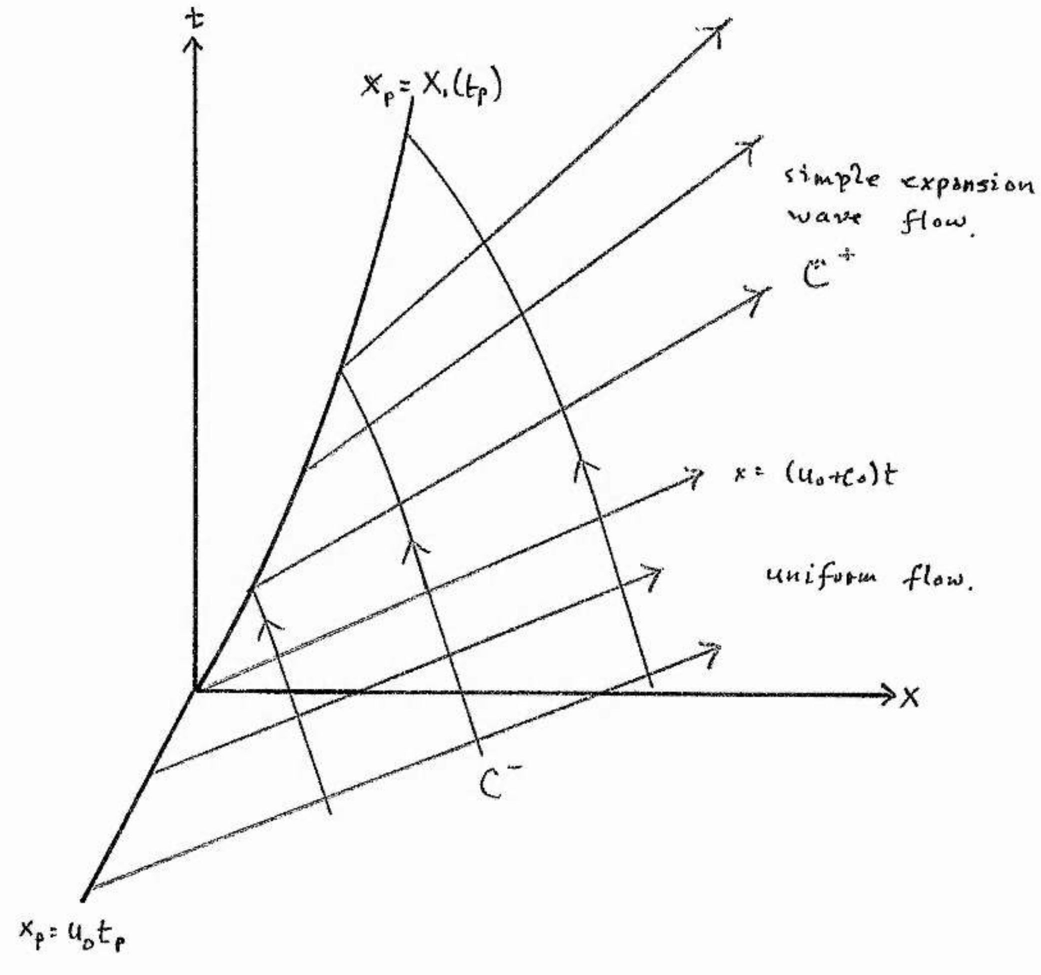


Figure 2



t
$x_p = X_1(t_p)$
simple expansion wave flow.
C^+
$x = (u_0+c_0)t$
uniform flow.
x
C^-
$x_p = u_0 t_p$

Figure 3

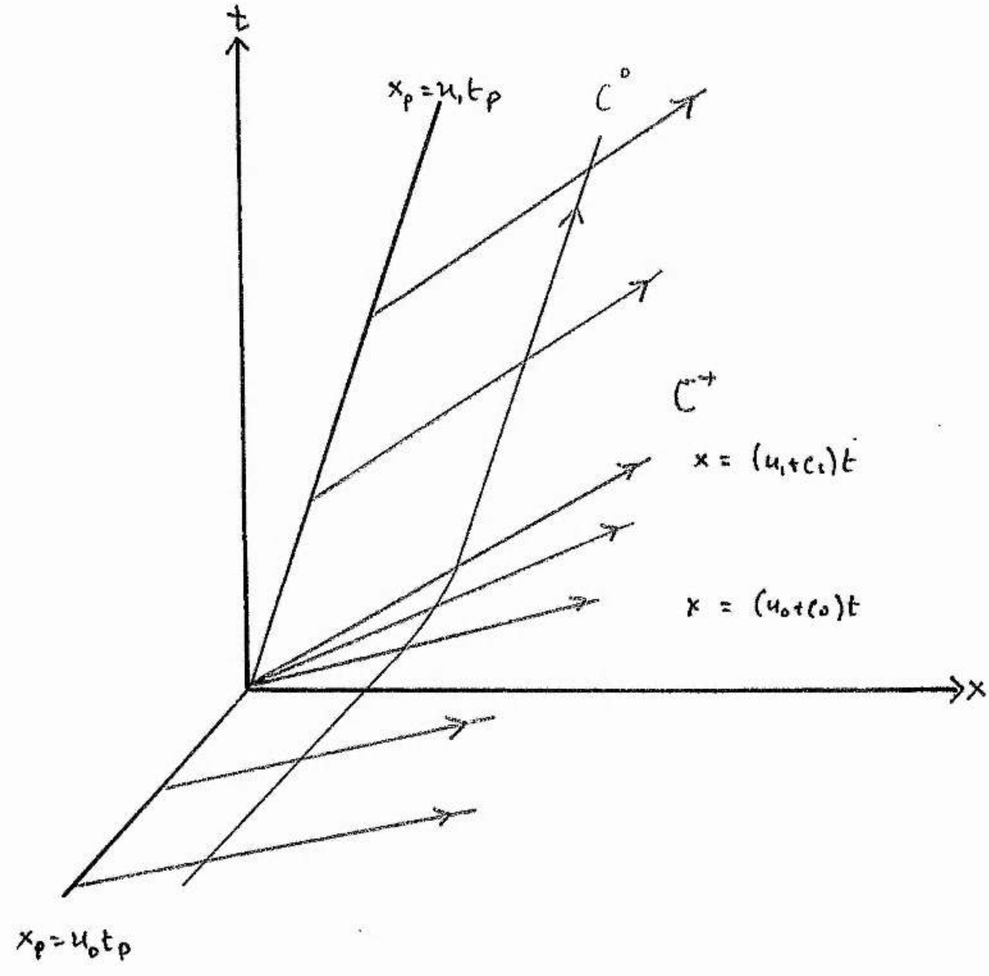


Figure 4

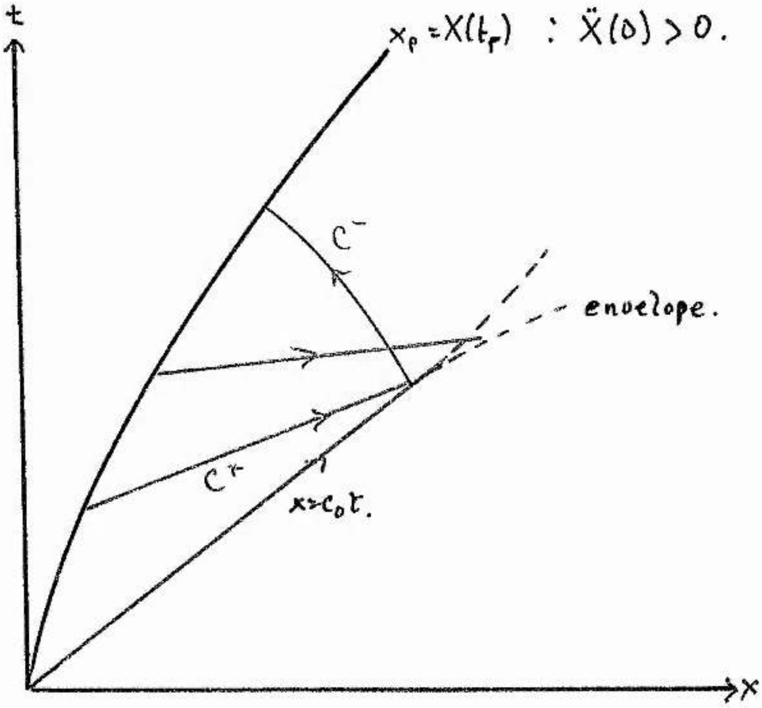


Figure 5

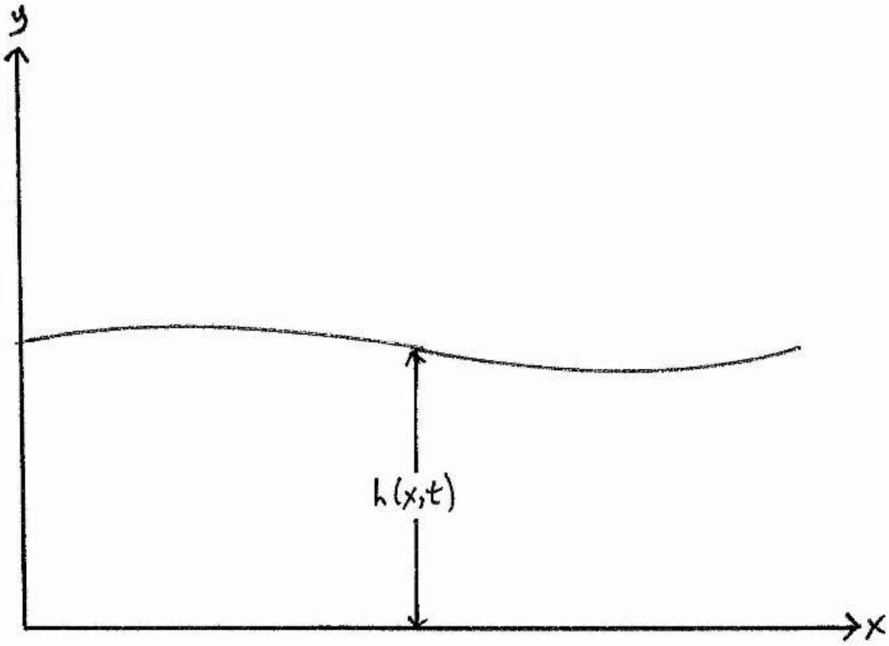


Figure 6

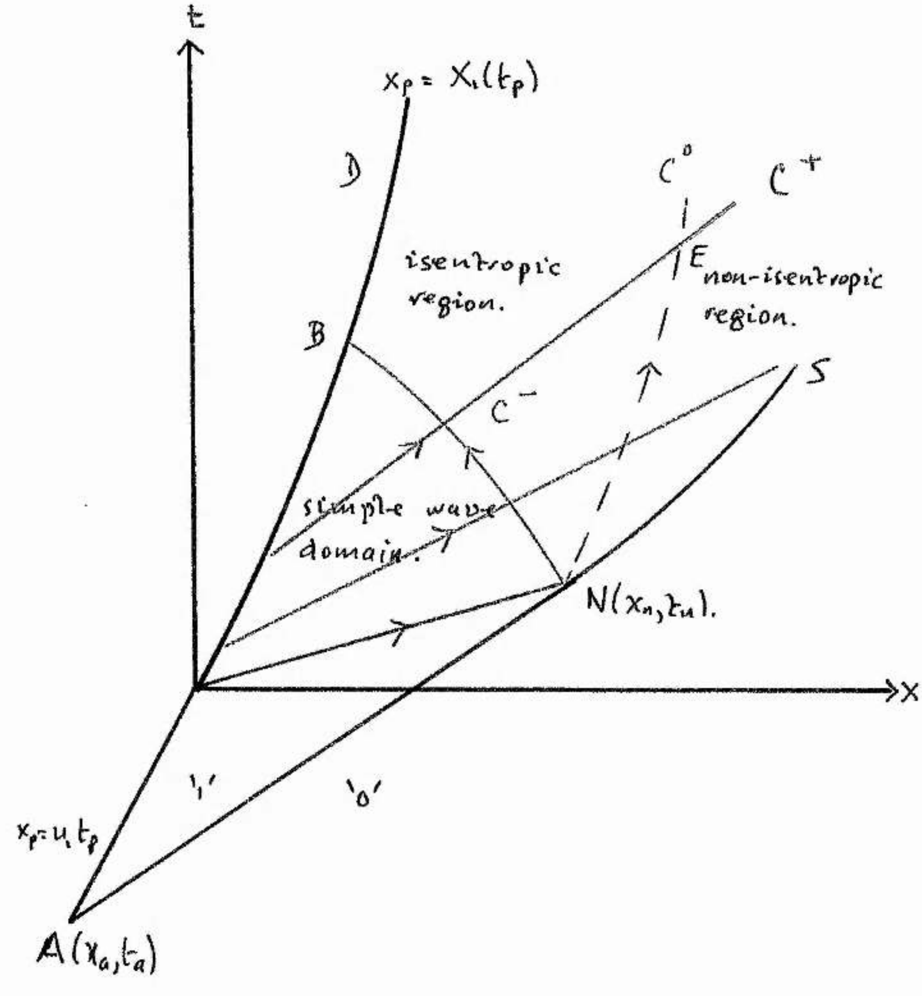

t
$x_p = X_1(t_p)$
D
C^0
C^+
isentropic region.
E
non-isentropic region.
B
S
C^-
simple wave domain.
$N(x_n, t_n)$.
x
'1'
'0'
$x_p = u_1 t_p$
$A(x_a, t_a)$

Figure 7

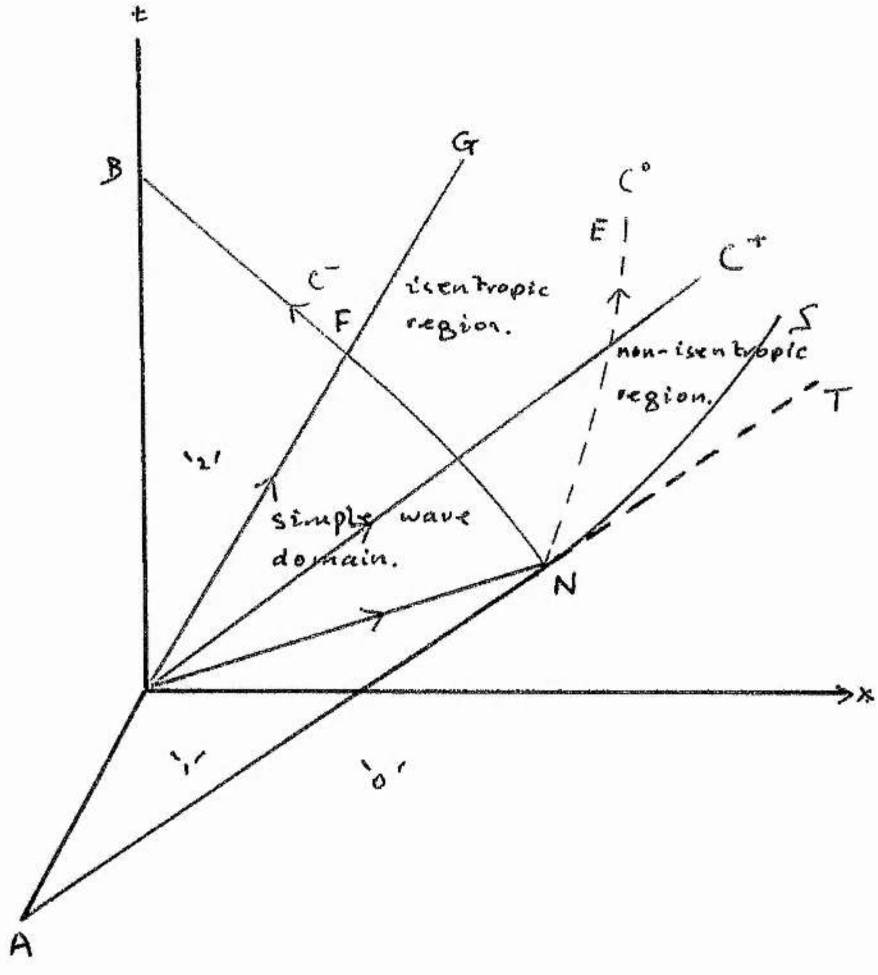

t
B
G
C°
E
C⁺
C⁻
isentropic
region.
F
non-isentropic
region.
S
T
'2'
simple wave
domain.
N
x
'1'
'0'
A

Figure 8

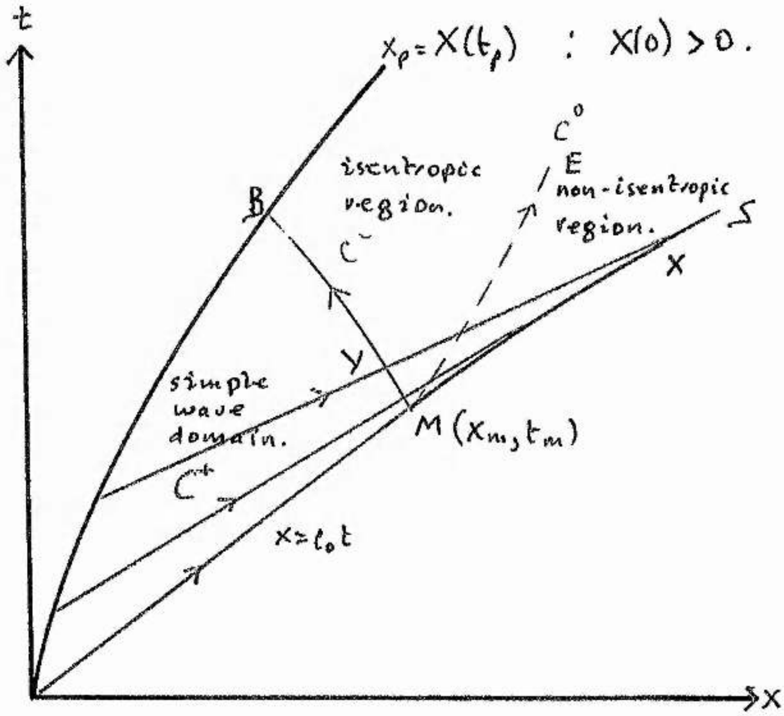

t
$x_p = X(t_p)$: $X(0) > 0$.
C^0
E
isentropic
region.
non-isentropic
region.
B
S
C^-
X
Y
simple
wave
domain.
$M(x_m, t_m)$
C^+
$x = c_0 t$
x

Figure 9

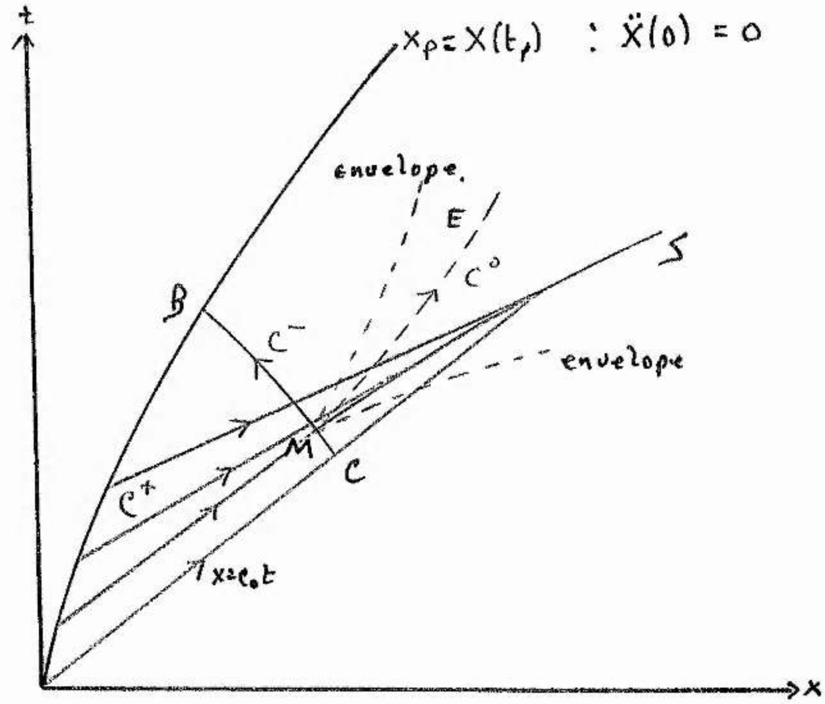


Figure 10

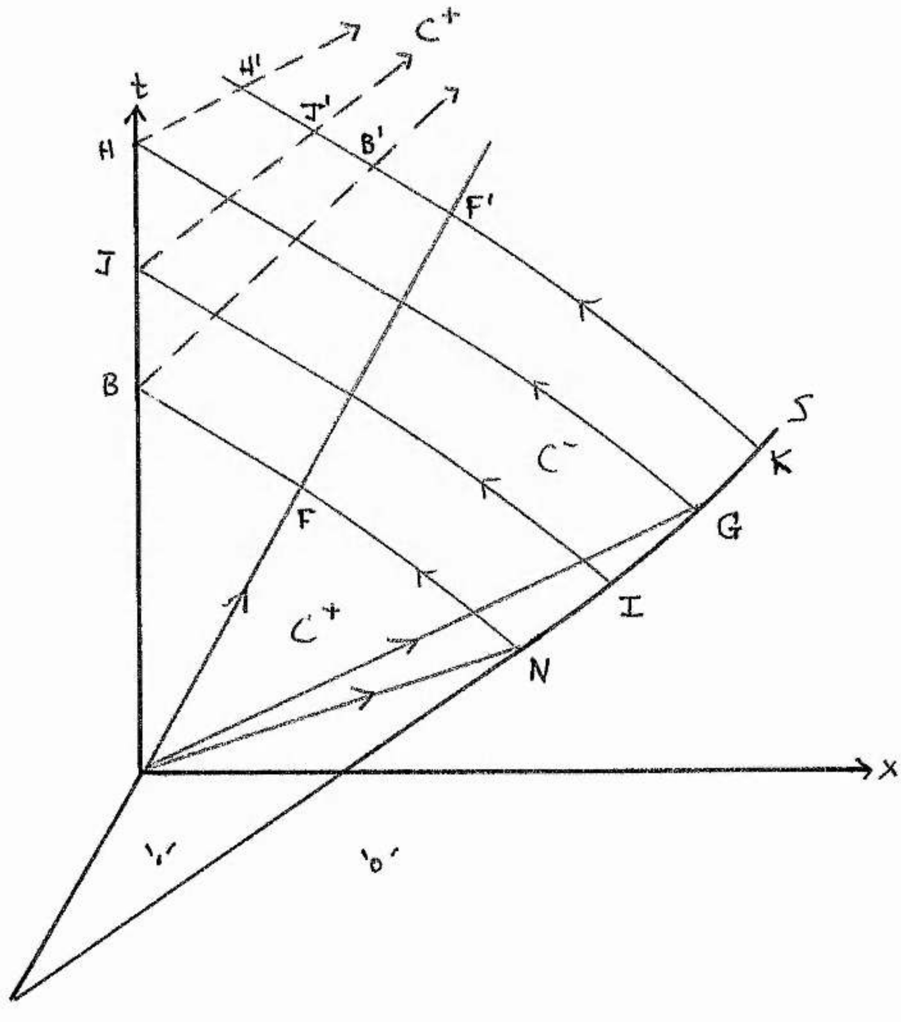


Figure 11

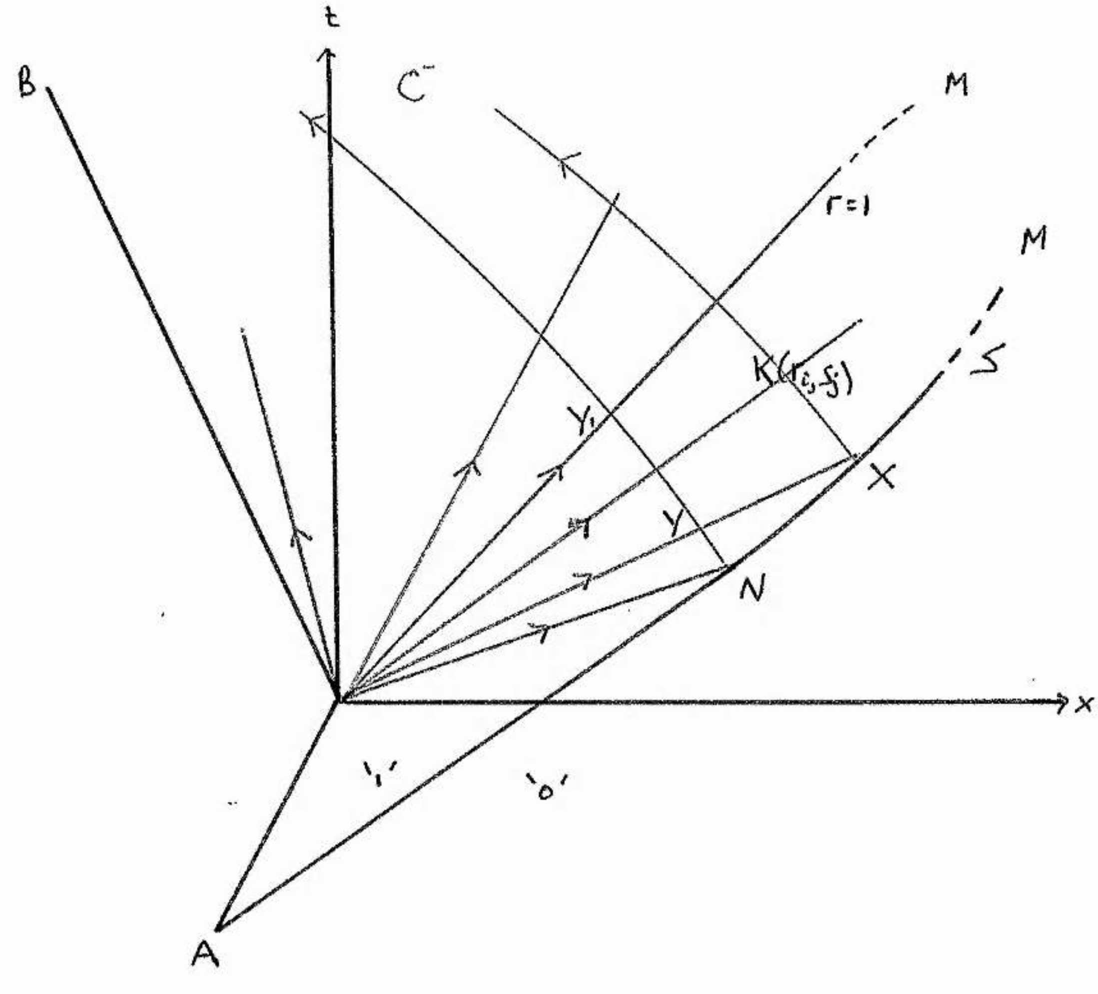


Figure 12

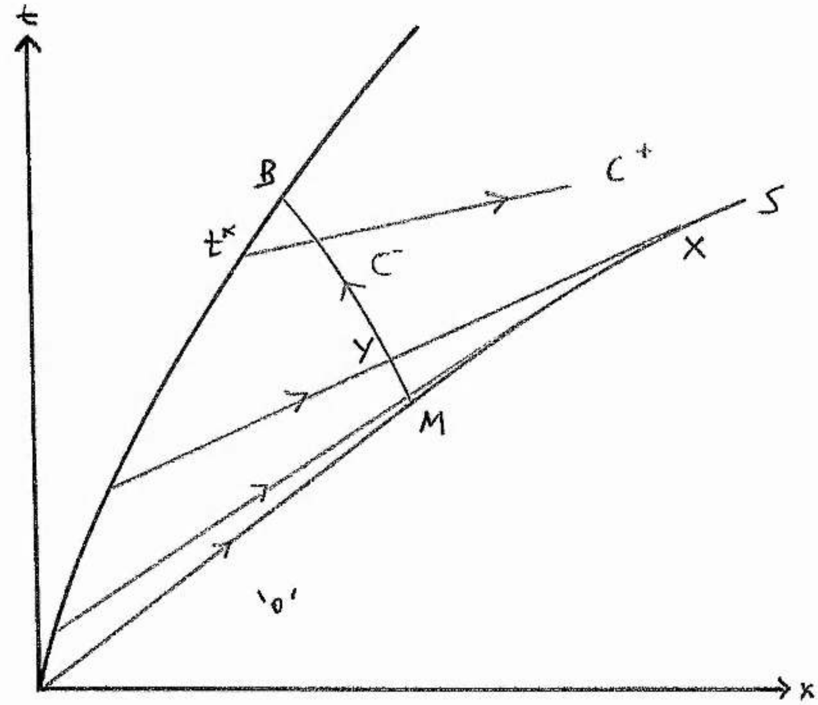


Figure 13

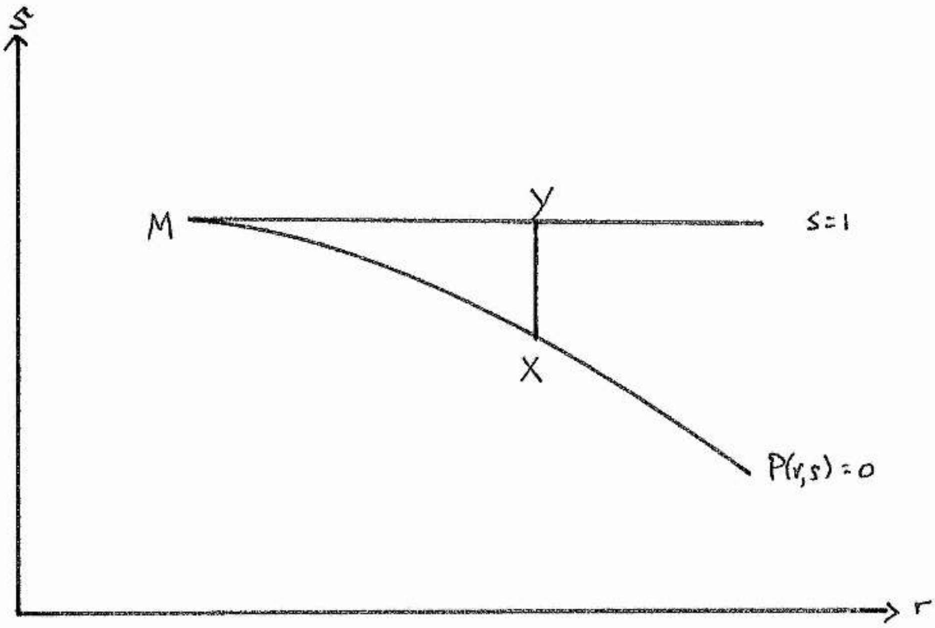


Figure 14

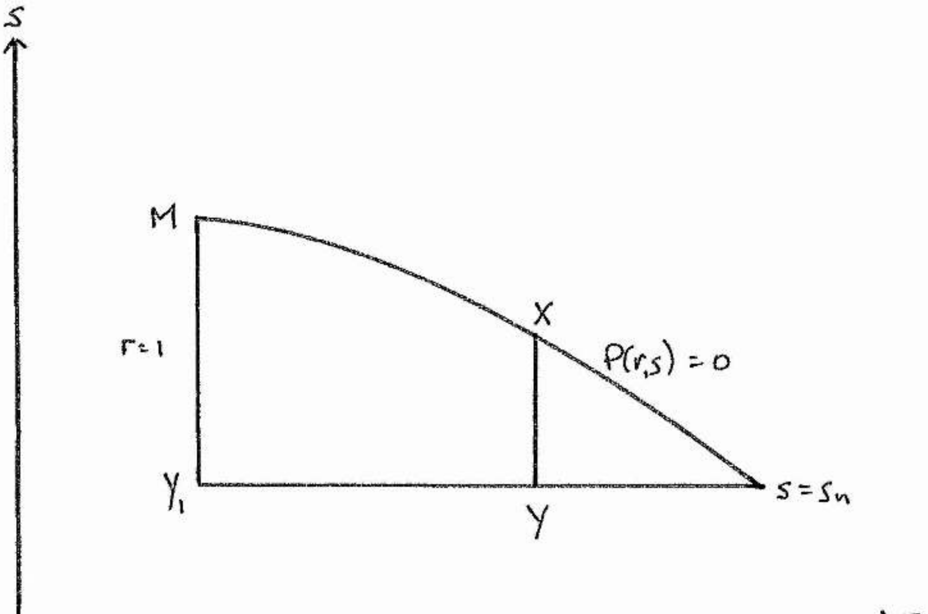


Figure 15

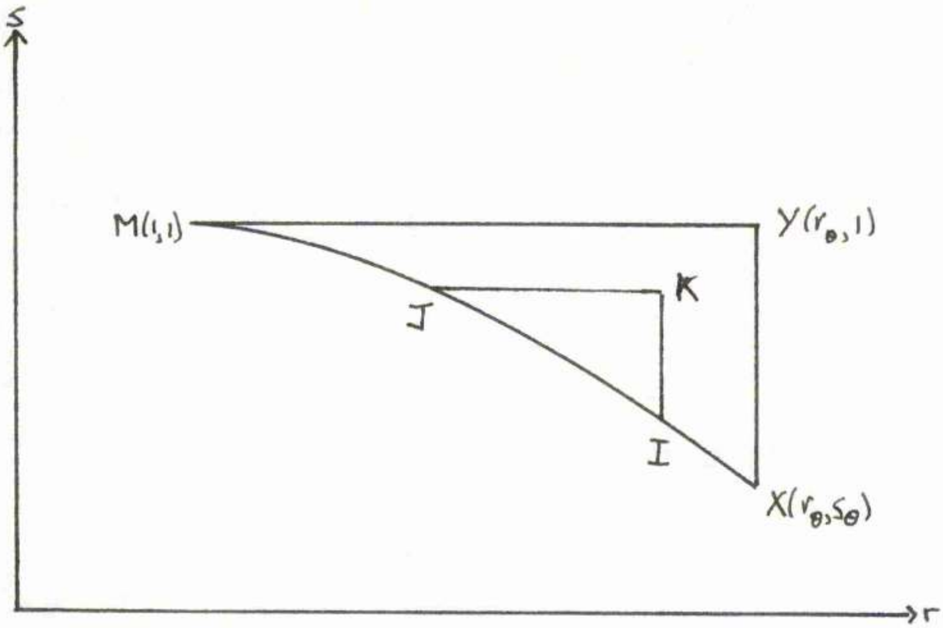


Figure 17

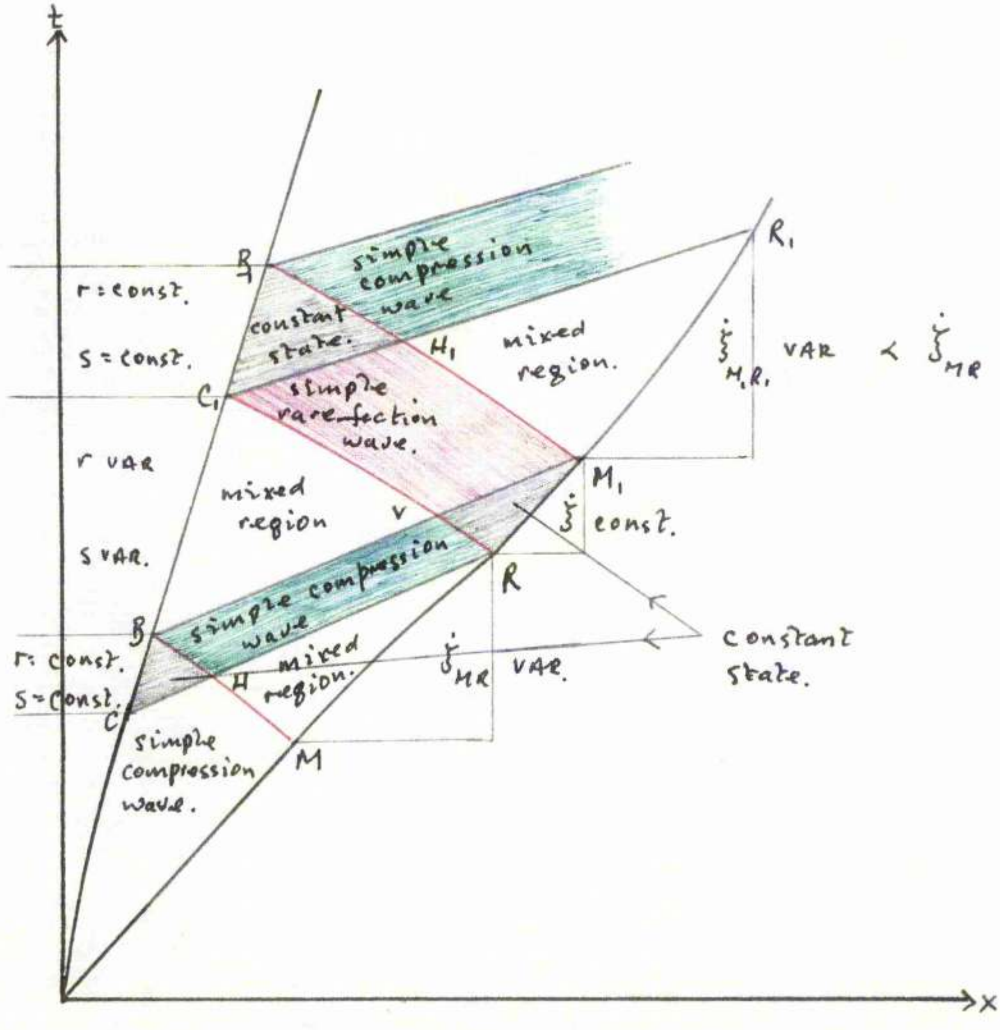


Figure 18

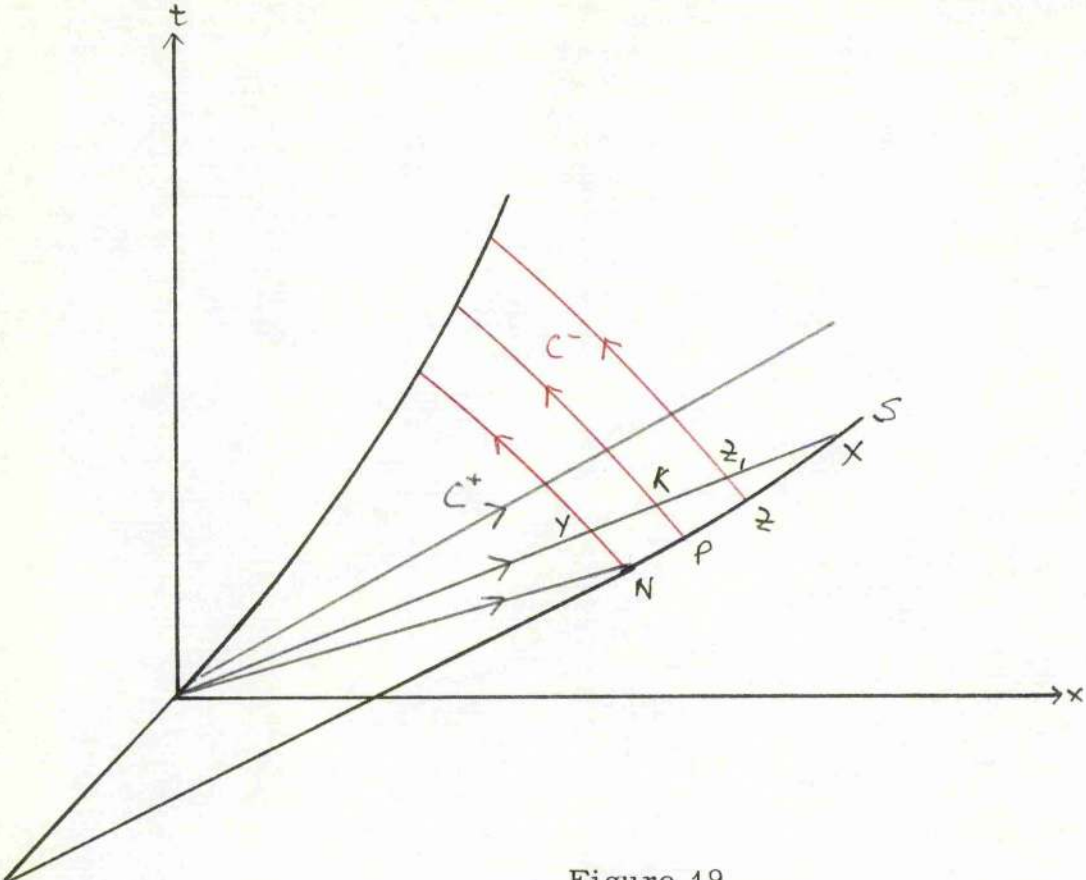


Figure 19

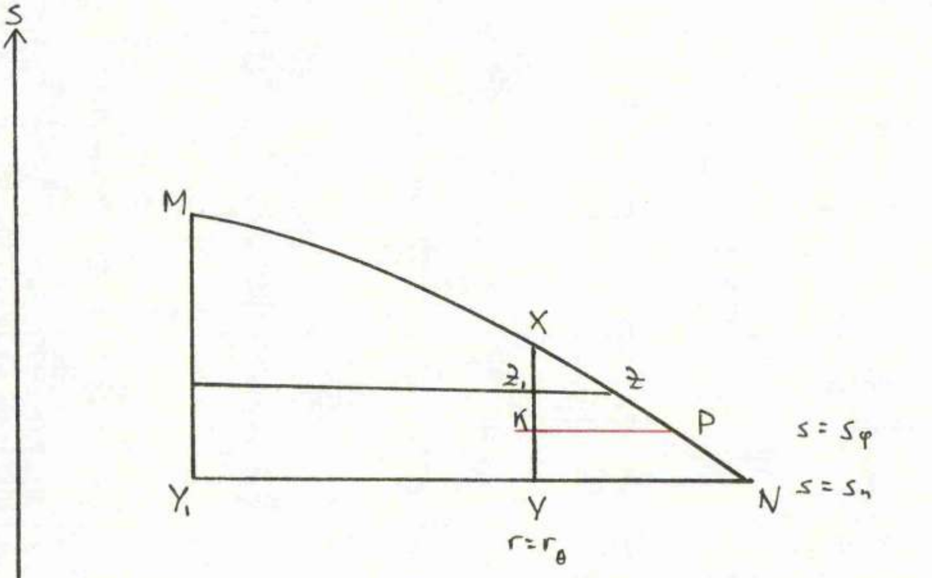


Figure 20

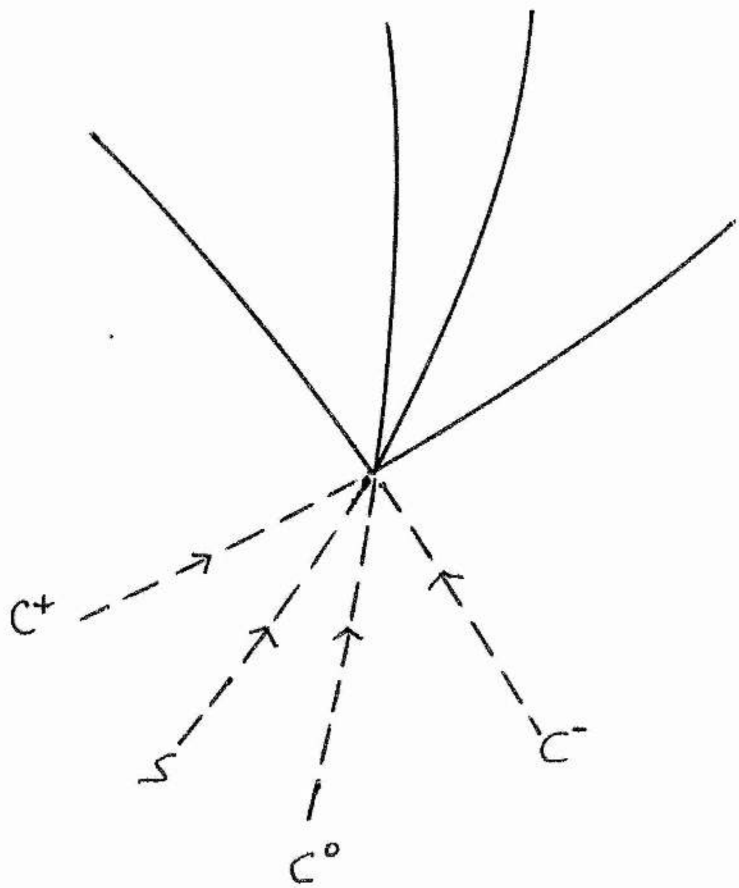


Figure 21

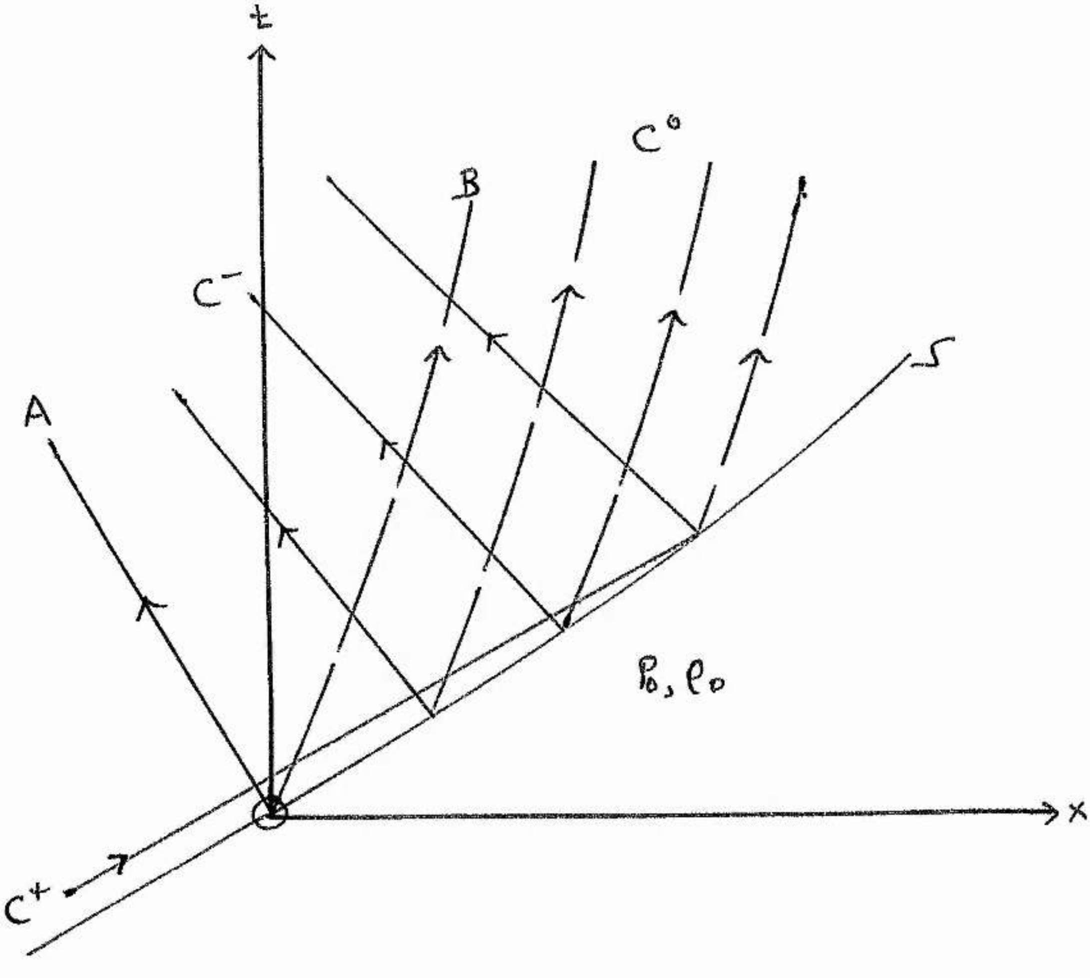


Figure 22

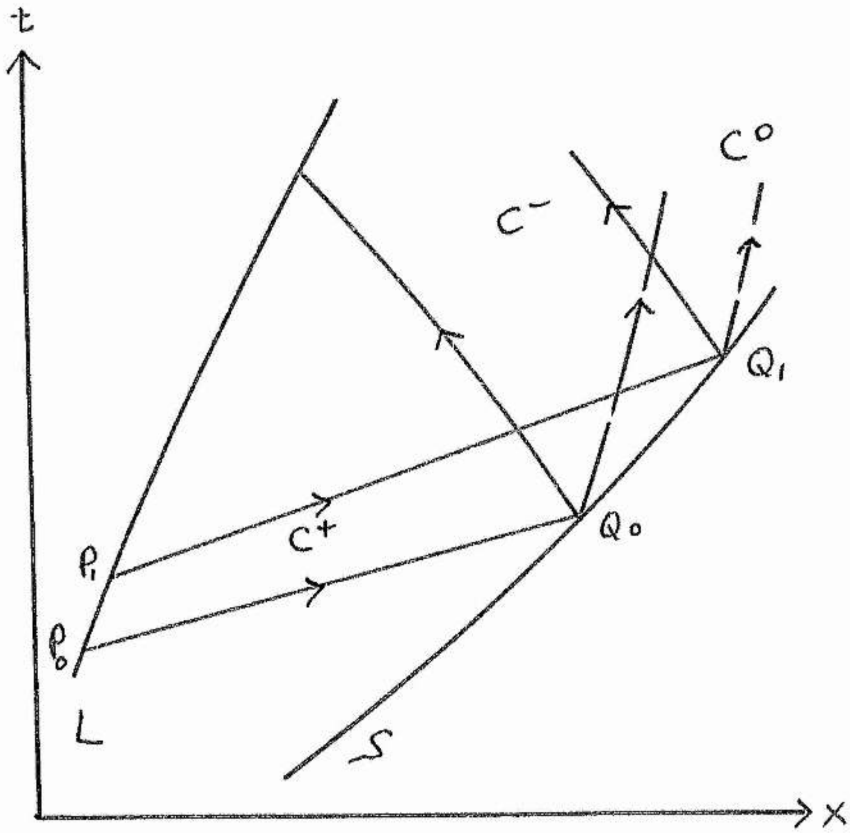


Figure 23

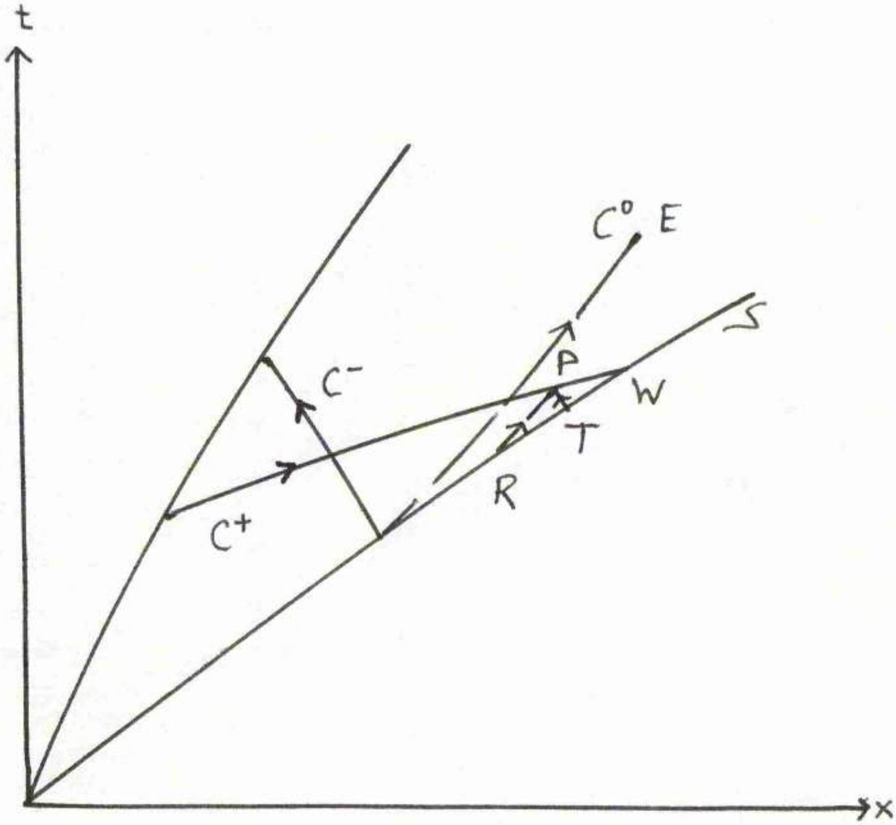


Figure 24

TABLE 1

σ	Δ_1	Δ_2	Percentage Difference
0	1	1	0
1/9	1.15,0	1.14,9	0.07
2/9	1.30,9	1.30,0	0.72
1/3	1.48,7	1.45,2	2.34
4/9	1.69,5	1.60,6	5.20
5/9	1.94,6	1.76,2	9.45
2/3	2.26,5	1.92,7	15.00

Δ_1, Δ_2 denote respectively the values of $(\frac{t}{t_n})$ as predicted by the present method and the 'simple wave' approximation.

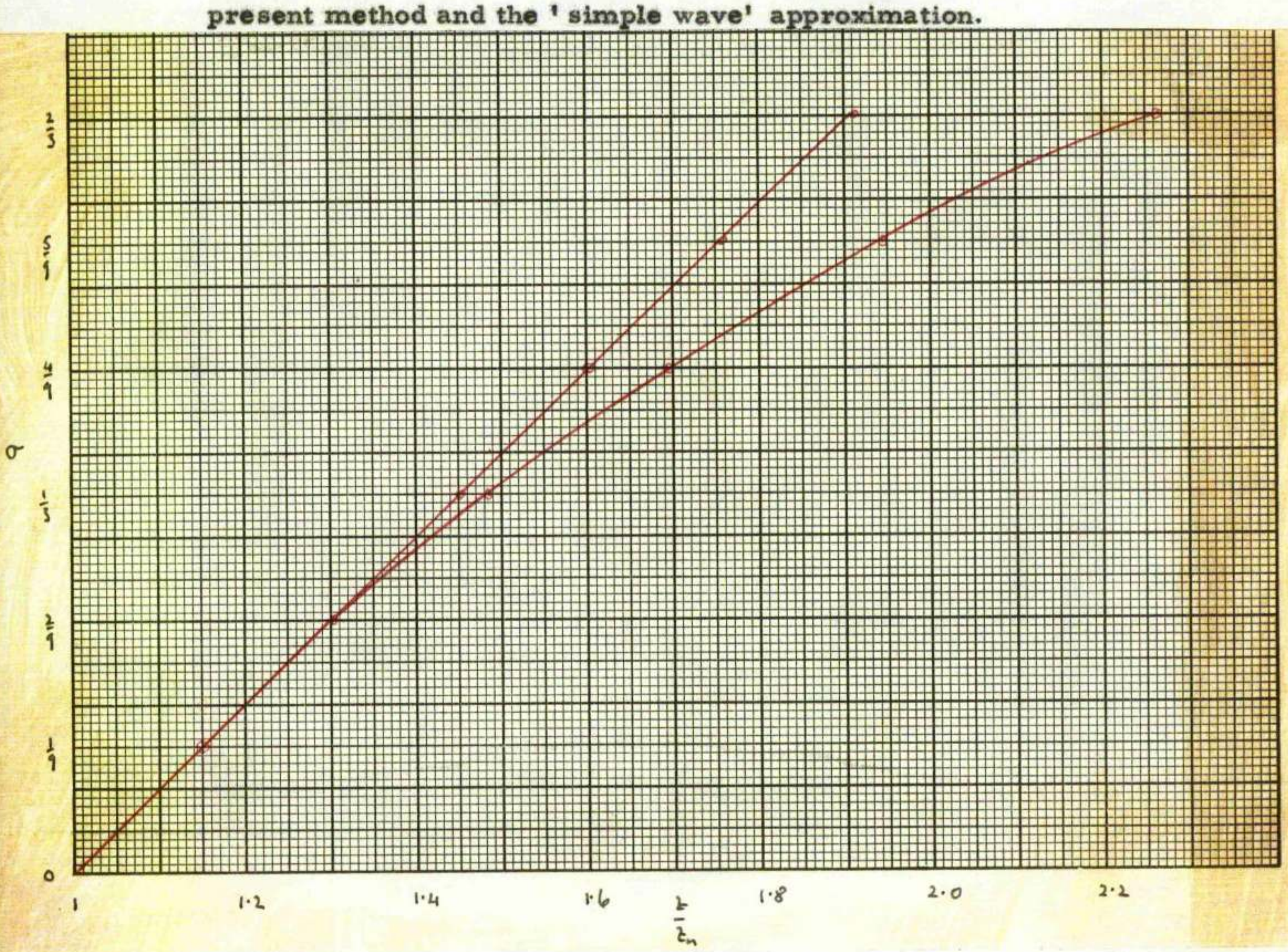


Figure 16

TABLE 2

$\dot{\xi}(N)$	$c_n^{\ 2}$	σ_n	D_1	D_2	D_3	Percentage Difference: Friedrichs	Percentage Difference: Int. Eqn.
1.149	1.2	0.191	10.86,7	10.87,2	10.89,8	0.28	0.05
1.369	1.5	0.454	4.80,4	4.80,8	4.86,7	1.31	0.08
1.587	1.8	0.698	3.26,3	3.26,6	3.34,2	2.42	0.09
2.019	2.5	1.224	2.01,5	2.02,7	2.13,8	6.12	0.60
2.449	3.0	1.577	1.63,8	1.66,1	1.78,3	8.86	1.41

D_1 : Exact value of $-\frac{1}{t_n}\left(\frac{dt}{dr}\right)$ from the integral equation.

D_2 : Value of $-\frac{1}{t_n}\left(\frac{dt}{dr}\right)$ from the present method derived from the integral equation.

D_3 : Significant value of $-\frac{1}{t_n}\left(\frac{dt}{dr}\right)_n$ from the 'simple wave' approximation.

TABLE 3

c_n^2	c^2	A	B	c_n^2	c^2	A	B
1.2	1.02	98.50	83.41	2.5	1.15	88.80	62.08
$\sigma_n = .191$	1.06	10.99	10.19	$\sigma_n = 1.224$	1.45	10.20	8.26
	1.10	3.97	3.72		1.60	3.75	3.34
	1.14	2.03	1.90		2.07	1.96	1.91
	1.18	1.23	2.22		2.35	1.22	1.20
1.5	1.05	96.31	77.67	3.0	1.2	85.00	60.17
$\sigma_n = .454$	1.15	10.81	9.08	$\sigma_n = 1.577$	1.6	9.89	8.27
	1.25	3.92	3.49		1.8	3.66	3.37
	1.35	2.02	1.89		2.4	1.93	1.88
	1.45	1.23	1.21		2.8	1.20	1.21
1.8	1.08	94.00	68.01				
$\sigma_n = .698$	1.24	10.62	8.65				
	1.40	3.88	3.41				
	1.56	2.00	1.87				
	1.72	1.23	1.20				

A : significant value of $(\frac{t}{t_n})$ obtained from the 'simple wave' approximation.

B : value of $(\frac{t}{t_n})$ obtained from the closed form of the 'simple wave' approximation.

TABLE 4

c_n^2	c^2	A	G	c_n^2	c^2	A	G
1.2	1.020	98.50	99.10	2.5	1.150	88.80	132.80
	1.022	79.78	80.27		1.667	72.01	107.65
	1.025	63.04	63.42		1.189	57.02	85.31
	1.033	35.49	35.71		1.250	32.29	48.06
	1.060	10.99	11.04		1.450	10.20	15.11
	1.100	3.97	3.98		1.600	3.75	5.34
	1.140	2.03	2.03		2.070	1.96	2.55
	1.180	1.23	1.23		2.350	1.22	1.38
	1.200	1	1		2.500	1	1
1.5	1.050	96.31	100.42	3.0	1.200	85.00	169.00
	1.055	78.60	81.32		1.222	69.01	137.05
	1.062	61.63	64.29		1.250	53.86	108.67
	1.083	34.78	36.25		1.333	31.02	61.19
	1.150	10.81	11.24		1.600	9.89	19.44
	1.250	3.92	4.05		1.800	3.66	6.85
	1.350	2.02	2.06		2.400	1.93	3.19
	1.450	1.23	1.24		2.800	1.20	1.51
	1.500	1	1		3.000	1	1
1.8	1.080	94.00	105.20				
	1.089	76.30	85.37				
	1.100	60.29	67.52				
	1.133	34.02	38.05				
	1.240	10.62	11.85				
	1.400	3.88	4.13				
	1.560	2.00	2.14				
	1.720	1.23	1.26				
	1.800	1	1				

A : significant value of $\left(\frac{t}{t_n}\right)$ obtained from the 'simple wave' approximation.

G : value of $\left(\frac{t}{t_n}\right)$ obtained by the present method.

TABLE 5

$k = 3$	π	μ_3	$\bar{\mu}_3$	$\mu_3 - \bar{\mu}_3$
	-0.20	0.539,3	0.470,9	0.068,4
	-0.15	0.548,8	0.480,2	0.068,6
	-0.10	0.560,5	0.491,8	0.068,7
	-0.05	0.573,6	0.504,7	0.068,9
	0	0.586,9	0.517,8	0.069,1
	0.05	0.601,6	0.532,3	0.069,3
	0.10	0.614,7	0.545,2	0.069,5
	0.15	0.626,4	0.556,8	0.069,6
	0.20	0.635,9	0.566,1	0.069,8
	0.25	0.642,4	0.572,4	0.070,0
	0.30	0.645,0	0.574,8	0.070,2

TABLE 6

P	δ	Δ_1	Δ_2	Δ_3
0	0	1	1	1
0.175	0.1	1.149,7	1.150,3	1.150,3
0.367	0.2	1.287,8	1.292,5	1.291,9
0.675	0.3	1.414,1	1.430,0	1.429,1
0.800	0.4	1.528,8	1.566,4	1.561,9
1.104	0.5	1.632,0	1.705,4	1.696,8
1.300	0.6	1.723,3	1.850,2	1.835,3
1.675	0.7	1.803,0	2.004,5	1.980,8

Δ_1, Δ_2 and Δ_3 denote respectively the values of $(\frac{t}{t_m})$ as predicted by the 'simple wave' approximation, the present method and shock-expansion theory.

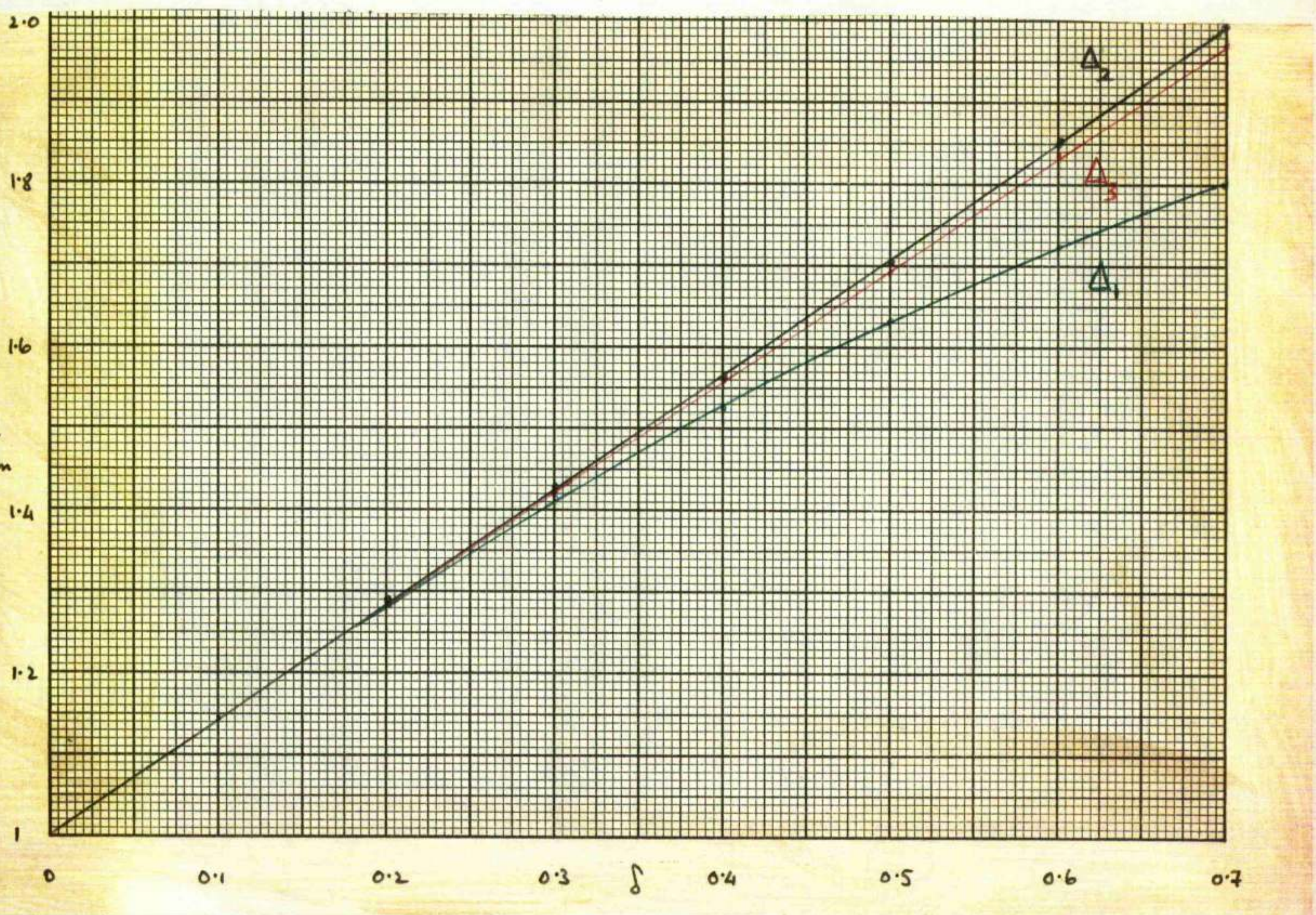


Figure 25

TABLE 7

P	δ	Δ_1	Δ_2	Δ_3	Δ_4
0	0	0	0	0	0
0.175	0.1	0.05	0.05,0	0.50,0	0.05,0
0.367	0.2	0.10	0.09,9	0.09,9	0.09,9
0.675	0.3	0.15	0.14,6	0.14,5	0.14,7
0.800	0.4	0.20	0.19,2	0.18,9	0.19,4
1.104	0.5	0.25	0.23,4	0.22,9	0.23,9
1.300	0.6	0.30	0.27,3	0.26,4	0.28,0
1.675	0.7	0.35	0.30,7	0.29,3	0.32,0

Δ_1, Δ_2, Δ_3 and Δ_4 denote respectively the values of $-\frac{t_n}{c_o}\left(\frac{d^2x}{dt^2}\right)_n$ as predicted by the 'simple wave' approximation, the present method, shock-expansion theory and first order perturbation theory.

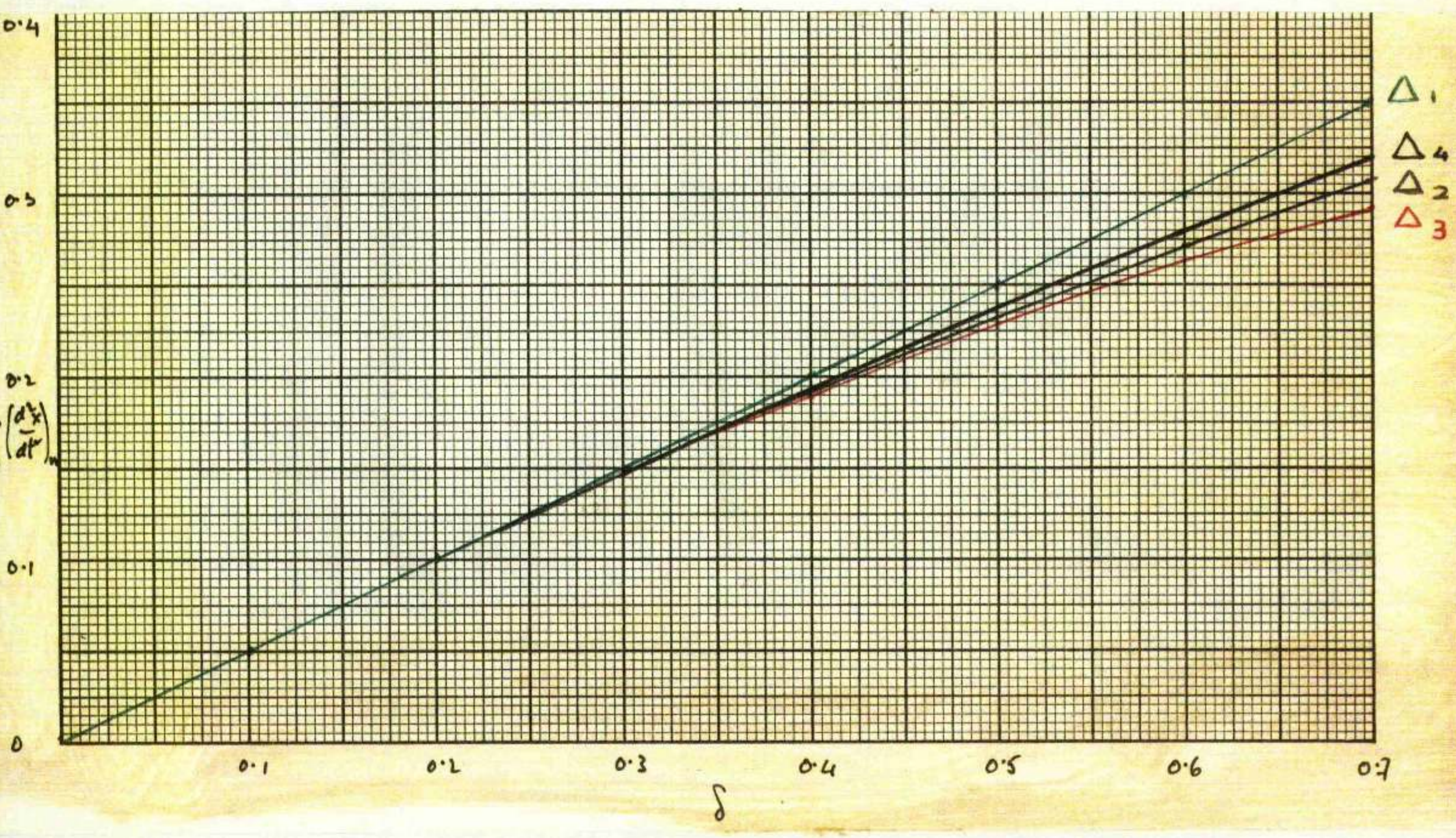


Figure 26

TABLE 8

P	δ	Δ_1	Δ_2	Δ_3
0	0	0	0	0
0.175	0.1	0.07,5	0.07,4	0.07,4
0.367	0.2	0.15,0	0.14,6	0.14,6
0.675	0.3	0.225	0.21,1	0.21,2
0.800	0.4	0.300	0.26,5	0.26,8
1.104	0.5	0.37,5	0.30,8	0.31,3
1.300	0.6	0.450	0.33,5	0.34,3
1.675	0.7	0.525	0.34,5	0.35,5

Δ_1, Δ_2 and Δ_3 denote respectively the values of $\frac{t_n^2}{c_o}\left(\frac{d^3x}{dt^3}\right)_n$ as predicted by the 'simple wave' approximation, the present method and shock-expansion theory.

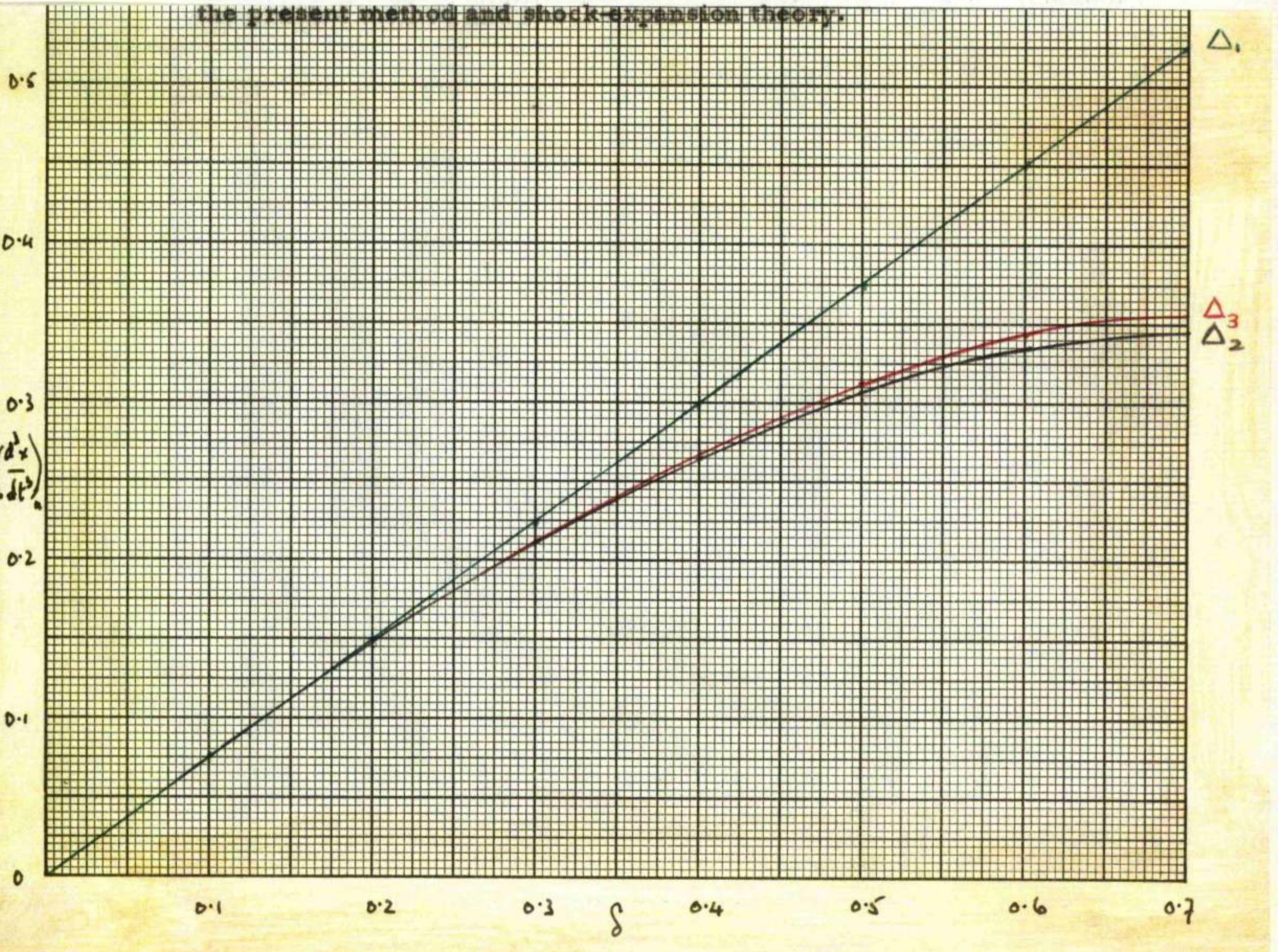


Figure 27

§A. (i) $\frac{ds}{dH} < 0$, $H > 1$; (ii) $s_n \le s \le 1$

(i) From (1.4.15), it follows that $s = \sqrt{H} - \frac{H-1}{2\sqrt{2}} \sqrt{\frac{1+H}{H}}$.

$$\therefore \frac{ds}{dH} = - \frac{[2H^2 + H + 1 - 2H\sqrt{2(H+1)}]}{4H\sqrt{2H(1+H)}}$$

$$= - \frac{[\sqrt{2}\,H - \sqrt{1+H}\,]^2}{4H\sqrt{2H(1+H)}} .$$

Hence $\frac{ds}{dH} < 0$ if $H > 1$. Also, it is evident that s is a monotonic decreasing function of H .

(ii) From (i), $s-1 = \frac{\sqrt{H} - 1}{2\sqrt{2H}} [2\sqrt{2H} - \sqrt{H+1}\,(1 + \sqrt{H})]$,

i.e. $s = 1$, if $H = 1$.

$s-1 < 0$ if $2\sqrt{2H} < \sqrt{H+1}\,(1+\sqrt{H})$.

Let $y = \sqrt{H} - 1$, then $y > 0$ if $H > 1$, and

$s-1 < 0$ if $2\sqrt{2}\,(1+y) < (y+2)\sqrt{2+2y+y^2}$

i.e. $s-1 < 0$ if $8(1+y)^2 < (y+2)^2\,(2+2y+y^2)$.

On collecting the terms in y , it is apparent that $s-1 < 0$ if

$$y^2(y^2 + 6y + 6) > 0 .$$

This inequality is satisfied since $y > 0$ and hence $s < 1$ if $H > 1$. Since s is a monotonic decreasing function of H , it follows that if $1 \le H \le H_n$ then $s_n \le s \le 1$.

§ B. The expansions (3.1.5) b.

To derive the required expansions it is convenient to introduce the parameter

$\theta = (\frac{r+s}{2})^2 - 1 = H - 1$.

Then, in terms of θ , equations (3.1.3) are

$$\dot{\xi} = \sqrt{(1+\theta)(1+\frac{\theta}{2})} \quad ,$$

$$\begin{matrix} r \\ s \end{matrix} = \sqrt{1+\theta} \pm \frac{\theta}{2} \sqrt{\frac{1+\frac{\theta}{2}}{1+\theta}} \quad .$$

Hence $$r = [1 + \theta - \frac{1}{4}\theta^2 + \frac{11}{64}\theta^3 - \frac{35}{256}\theta^4 + \frac{475}{4096}\theta^5 + 0(\theta^6)] \quad ,$$

$$s = [1 - \frac{3}{64}\theta^3 + \frac{15}{256}\theta^4 - \frac{251}{4096}\theta^5 + 0(\theta^6)] \quad .$$

$$\therefore \frac{2}{3}(\frac{3}{2}r - \frac{1}{2}s - 1) = \sigma = \theta - \frac{1}{4}\theta^2 + \frac{3}{16}\theta^3 - \frac{5}{32}\theta^4 + \frac{419}{3072}\theta^5 + 0(\theta^6) \quad ,$$

$$\theta = \sigma + \frac{1}{4}\sigma^2 - \frac{1}{16}\sigma^3 + \frac{37}{3072}\sigma^5 + 0(\sigma^6) \quad .$$

Consequently,

$$\dot{\xi} = 1 + \frac{3}{4}\sigma + \frac{5}{32}\sigma^2 - \frac{5}{128}\sigma^3 + \frac{3}{2048}\sigma^4 + \frac{51}{8192}\sigma^5 + 0(\sigma^6) \quad ,$$

$$r = 1 + \sigma - \frac{1}{64}\sigma^3 + \frac{1}{128}\sigma^4 - \frac{11}{12,288}\sigma^5 + 0(\sigma^6) \quad ,$$

$$s = 1 - \frac{3}{64}\sigma^3 + \frac{3}{128}\sigma^4 - \frac{11}{4096}\sigma^5 + 0(\sigma^6) \quad .$$

§C. The expansions (3.4.12).

From §B, the following series expansions are obtained:

$$\text{(i)}\quad (1+r)^{-1} = \frac{1}{2}\left[1 - \frac{1}{2}\sigma + \frac{1}{4}\sigma^2 - \frac{15}{128}\sigma^3 + \frac{13}{256}\sigma^4 + 0(\sigma^5)\right],$$

$$(1+r)^{\frac{1}{2}} = \sqrt{2}\left[1 + \frac{1}{4}\sigma - \frac{1}{32}\sigma^2 + \frac{1}{256}\sigma^3 + \frac{1}{2048}\sigma^4 + 0(\sigma^5)\right],$$

$$(1+r)^{-\frac{3}{2}} = \frac{1}{2\sqrt{2}}\left[1 - \frac{3}{4}\sigma + \frac{15}{32}\sigma^2 - \frac{67}{256}\sigma^3 + \frac{273}{2048}\sigma^4 + 0(\sigma^5)\right],$$

$$(1+r_\theta)^{-2} = \frac{1}{4}\left[1 - \epsilon + \frac{3}{4}\epsilon^2 - \frac{31}{64}\epsilon^3 + \frac{9}{32}\epsilon^4 + 0(\epsilon^5)\right],$$

$$(r_\theta+s) = 2\left[1 + \frac{\epsilon}{2} - \frac{\epsilon^3+3\sigma^3}{128} + \frac{\epsilon^4+3\sigma^4}{256} + 0(\sigma^5)\right],$$

$$(r_\theta+s)^{-1} = \frac{1}{2}\left[1 - \frac{\epsilon}{2} + \frac{\epsilon^2}{4} - \frac{15\epsilon^3-3\sigma^3}{128} + \frac{13\epsilon^4-6\epsilon\sigma^3-3\sigma^4}{256} + 0(\sigma^5)\right],$$

$$(r_\theta+s)^{\frac{1}{2}} = \sqrt{2}\left[1 + \frac{\epsilon}{4} - \frac{\epsilon^2}{32} + \frac{\epsilon^3-3\sigma^2}{256} + \frac{\epsilon^4+6\epsilon\sigma^3+12\sigma^4}{2048} + 0(\sigma^5)\right].$$

From (3.1.7)a and (i), it follows that (3.4.12)a is given by

$$\text{(ii)}\quad \frac{ds}{dr} = -\frac{9}{64}\sigma^2\left[1 - \frac{2}{3}\sigma + \frac{49}{288}\sigma^2\right] + 0(\sigma^5).$$

From (3.1.8), $\dfrac{d\dot{\xi}}{dr} = \dfrac{1}{2\sqrt{2}}\left[1 + \dfrac{ds}{dr}\right]\left[(r+s)^2+2\right]\left[(r+s)^2+4\right]^{-\frac{1}{2}}$

and consequently (3.4.12)b is given by

$$\text{(iii)}\quad \frac{d\dot{\xi}}{dr} = \frac{3}{4}\left[1 + \frac{5}{12}\sigma - \frac{7}{64}\sigma^2 - \frac{1}{256}\sigma^3 + \frac{57}{2048}\sigma^4\right] + 0(\sigma^5).$$

By definition,

$$w(r, s, r_\theta, 1) = (r+1)^{\frac{1}{2}}(r_\theta+s)^{\frac{1}{2}}(1+r_\theta)^{-2}\,F\left(-\frac{1}{2}, -\frac{1}{2}, 1; p\right),$$

where $\quad p = (r-r_\theta)(s-1)(1-r)^{-1}(r_\theta+s)^{-1}$

From (i), we obtain:

(iv) $$(r+1)^{\frac{1}{2}}(r_\theta+s)^{\frac{1}{2}}(1+r_\theta)^{-2}=\frac{1}{2}\left[1+\frac{\sigma-3\epsilon}{4}+\frac{15\epsilon^2-6\epsilon\sigma-\sigma^2}{32}-\frac{67\epsilon^3-30\epsilon^2\sigma-6\epsilon\sigma^2+2\sigma^3}{256}+\frac{546\epsilon^4-268\epsilon^3\sigma-63\epsilon^2\sigma^2+48\epsilon\sigma^3+7\sigma^4}{4096}\right]+0(\sigma^5),$$

(v) $$p=-\frac{3}{256}\sigma^3(\sigma-\epsilon)+0(\sigma^5).$$

Also, $F(-\frac{1}{2}, -\frac{1}{2}, 1; p)=1+\frac{1}{4}p+0(p^2)=1-\frac{3}{1024}\sigma^3(\sigma-\epsilon)+0(\sigma^5)$.

It then follows that (3.4.12) is given by

$$w(r,s,r_\theta,1)=\frac{1}{2}\left[1+\frac{\sigma-3\epsilon}{4}+\frac{15\epsilon^2-6\epsilon\sigma-\sigma^2}{32}-\frac{67\epsilon^3-30\epsilon^2\sigma-6\epsilon\sigma^2+2\sigma^3}{256}+\frac{546\epsilon^4-286\epsilon^3\sigma-63\epsilon^2\sigma^2+60\epsilon\sigma^3-5\sigma^4}{4096}\right]+0(\sigma^5).$$

Now, $$\frac{\partial w}{\partial s}=\frac{w}{(r_\theta+s)}\left[\frac{1}{2}+\frac{(1+r_\theta)(r-r_\theta)}{(s+r_\theta)(1+r)}\frac{d\log F}{dp}\right]$$

and $$\frac{\partial w}{\partial r}=\frac{w}{(r+1)}\left[\frac{1}{2}+\frac{(s-1)(1+r_\theta)}{(s+r_\theta)(1+r)}\frac{d\log F}{dp}\right],$$

where $$\frac{d\log F}{dp}=\frac{1}{4}\left[1-\frac{1}{8}p+0(p^2)\right]=\frac{1}{4}\left[1+\frac{3}{2048}\sigma^3(\sigma-\epsilon)\right]+0(\sigma^5).$$

From (i): $$\frac{(1+r_\theta)(r-r_\theta)}{(s+r_\theta)(1+r)}=\frac{1}{2}(\sigma-\epsilon)\left[1-\frac{\sigma}{2}+\frac{15\sigma^2-\sigma\epsilon-\epsilon^2}{64}+\frac{\epsilon^3+2\epsilon^2\sigma+2\epsilon\sigma^2-10\sigma^3}{128}\right]+0(\sigma^5),$$

$$\frac{(s-1)(1+r_\theta)}{(s+r_\theta)(1+r)}=-\frac{3}{128}\sigma^3[1-\sigma]+0(\sigma^5).$$

$$\therefore\quad\frac{\partial w}{\partial s}=\frac{w}{2(r_\theta+s)}\left[1+\frac{\sigma-\epsilon}{4}\left\{1-\frac{\sigma}{2}+\frac{15\sigma^2-\sigma\epsilon-\epsilon^2}{64}+\frac{\epsilon^3+2\epsilon^2\sigma+2\epsilon\sigma^2-10\sigma^3}{128}\right\}\right]+0(\sigma^5)$$

and $$\frac{\partial w}{\partial r}=\frac{w}{2(r+1)}\left[1-\frac{3}{256}\sigma^3(1-\sigma)\right]+0(\sigma^5).$$

(vi) From §B: $\frac{3}{2}r - \frac{1}{2}s - \dot{\xi} = \frac{3}{4}\sigma[1 - \frac{5}{24}\sigma + \frac{5}{96}\sigma^2 - \frac{1}{512}\sigma^3] + 0(\sigma^5)$

$$\frac{1}{2}r - \frac{3}{2}s - \dot{\xi} = -2[1 + \frac{1}{8}\sigma + \frac{5}{64}\sigma^2 - \frac{13}{256}\sigma^3 + \frac{61}{4096}\sigma^4] + 0(\sigma^5) \ .$$

(vii) Hence (3.4.12)c is given by

$$[(\frac{3}{2}r-\frac{1}{2}s-\dot{\xi})w_r+(\frac{1}{2}r-\frac{3}{2}s-\dot{\xi})w_s\frac{ds}{dr}] = \frac{3}{16}w\sigma[1-\frac{\sigma}{3}+\frac{19\sigma^2-18\sigma\epsilon}{64} - \frac{296\sigma^3-234\sigma^2\epsilon-216\sigma\epsilon^2}{1536}] + 0(\sigma^5) \ .$$

The series expansion of the kernal of (3.5.8)

Since the expansion of $k_1(r_\theta, s_n, r, s)$ is considered only to terms of order σ^3, s_n may be replaced by unity whenever it occurs with terms of order σ and higher.

$$w(r_\theta, s_n, r, s) = (r+s_n)^{\frac{1}{2}}(r_\theta+s)^{\frac{1}{2}}(r_\theta+s_n)^{-2}F(-\frac{1}{2}, -\frac{1}{2}, 1; p) \ ,$$

$$p = (r-r_\theta)(s-s_n)(r+s_n)^{-1}(r_\theta+s)^{-1} \ .$$

From (iii) it follows that

$$\text{(viii)} \quad (r+s_n)^{\frac{1}{2}}(r_\theta+s)^{\frac{1}{2}}(r_\theta+s_n)^{-2} = \sqrt{2}(1+s_n)^{-\frac{3}{2}}[1+\frac{\sigma-3\epsilon}{4}+\frac{15\epsilon^2-6\epsilon\sigma-\sigma^2}{32} - \frac{67\epsilon^3-30\epsilon^2\sigma-6\epsilon\sigma^2+2\sigma^3}{256}] + 0(\sigma^4) \ .$$

Also, from (v), $p = 0(\sigma^4)$

and hence $F(p) = 1 + 0(\sigma^4)$.

Consequently $w(r_\theta, s_n, r, s)$ is as given by (viii) .

$$\text{Now, } \frac{\partial w}{\partial s} = \frac{w}{(r_\theta+s)}[\frac{1}{2}+\frac{(r_\theta+s_n)(r-r_\theta)}{(r+s_n)(r_\theta+s)}\frac{dF}{dp}]$$

$$\text{and } \frac{\partial w}{\partial r} = \frac{w}{(r+s_n)}[\frac{1}{2}+\frac{(r_\theta+s_n)(s-s_n)}{(r+s_n)(r_\theta+s)}\frac{dF}{dp}] \ ,$$

where $\dfrac{dF}{dp} = \dfrac{1}{4} + 0(\sigma^4)$.

Consequently, $\dfrac{\partial w}{\partial s} = \dfrac{w}{2(r_\theta + s)}\left[1 + \dfrac{\sigma - \epsilon}{4}\left\{1 - \dfrac{\sigma}{2} + \dfrac{15\sigma^2 - \sigma\epsilon - \epsilon^2}{64}\right\}\right] + 0(\sigma^4)$,

and $\dfrac{\partial w}{\partial r} = \dfrac{w}{2(r + s_n)}[1 + 0(\sigma^4)]$.

The term $[(\frac{3}{2}r - \frac{1}{2}s - \dot{\xi})w_r + (\frac{1}{2}r - \frac{3}{2}s - \dot{\xi})w_s \frac{ds}{dr}]$ is then as determined by (vii) with w given by (vii). $\dfrac{d\dot{\xi}}{dr}$, $\dfrac{dr}{d\sigma}$ are given to terms of order σ^4 by the expansions (ii), (3.4.12)d. On substituting for the various terms their respective expansions noted above, we obtain, after some manipulation, the required expansion for $k_1(r_\theta, s_n, r, s_n)\dfrac{dr}{d\sigma}$,

$$k_1(r_\theta, s_n, r, s_n)\frac{dr}{d\sigma} = \frac{3}{2\sqrt{2}}(1+s_n)^{-\frac{3}{2}}\left[1 + \frac{5\sigma - 9\epsilon}{12} - \frac{2\sigma^2 + 10\sigma\epsilon - 15\epsilon^2}{32} - \frac{22\sigma^3 - 30\sigma^2\epsilon - 50\sigma\epsilon^2 + 67\epsilon^3}{256}\right] + 0(\sigma^4) .$$

The expansion (3.5.9)

To terms of order σ^4,

$$(r+s_n)^{\frac{3}{2}} = \frac{(1+s_n)^{\frac{3}{2}}}{2\sqrt{2}}(1+r)^{\frac{3}{2}} ,$$

i.e. $(r+s_n)^{\frac{3}{2}} = (1+s_n)^{\frac{3}{2}}[1 + \frac{3}{4}\sigma + \frac{3}{32}\sigma^2 - \frac{5}{256}\sigma^3] + 0(\sigma^4)$.

Also from (i), (vi) respectively,

$$(r+s)^{-\frac{1}{2}} = \frac{1}{\sqrt{2}}[1 - \frac{\sigma}{4} + \frac{3}{32}\sigma^2 - \frac{3}{128}\sigma^3] + 0(\sigma^4) .$$

$$(\frac{3}{2}r - \frac{1}{2}s - \dot{\xi})^{-1} = \frac{4}{3\sigma}[1 + \frac{5}{24}\sigma - \frac{5}{576}\sigma^2 - \frac{37}{3456}\sigma^3] + 0(\sigma^4) ,$$

$$\therefore \quad \frac{(r+s)^{-\frac{1}{2}}(r+s_n)^{\frac{3}{2}}}{(\frac{3}{2}r - \frac{1}{2}s - \dot{\xi})} = \frac{2\sqrt{2}(1+s_n)^{\frac{3}{2}}}{3\sigma}\left[\frac{17}{24}\sigma + \frac{55}{576}\sigma^2 - \frac{77}{6912}\sigma^3\right] + 0(\sigma^4) .$$

§ D. $W(r, s, r_\theta, s_n) = (r+s)\, w(r, s, r_\theta, s_n)$

For the decay problem, the classical Riemann function of (3.4.1) is given by (3.4.3), i.e.

$$W = \left(\frac{r+s}{r_\theta + s_n}\right)^{\frac{3}{2}} F\left(-\frac{1}{2}, \frac{3}{2}, 1 ; z\right)$$

where $$z = (r-r_\theta)(s_n - s)(r+s)^{-1}(r_\theta + s_n)^{-1} .$$

From the theory of the hypergeometric function, it may be shown that

$$F(\alpha, \beta, \gamma; x) = (1-x)^{-\alpha} F\left(\alpha, \gamma-\beta, \gamma; \frac{x}{x-1}\right) ,$$

that is, $$F\left(-\frac{1}{2}, \frac{3}{2}, 1; z\right) = (1-z)^{\frac{1}{2}} F\left(-\frac{1}{2}, -\frac{1}{2}, 1 ; \frac{z}{z-1}\right) .$$

But, $$\frac{z}{z-1} = \frac{(r-r_\theta)(s-s_n)}{(r+s_n)(r_\theta+s)} .$$

$$\therefore \quad F\left(-\frac{1}{2}, \frac{3}{2}, 1; x\right) = \frac{(r+s_n)^{\frac{1}{2}}(r_\theta+s)^{\frac{1}{2}}}{(r+s)^{\frac{1}{2}}(r_\theta+s_n)^{\frac{1}{2}}} F\left(-\frac{1}{2}, -\frac{1}{2}, 1; p\right) ,$$

where $$p = \frac{(r-r_\theta)(s-s_n)}{(r+s_n)(r_\theta+s)} .$$

Hence $$W = \left(\frac{r+s}{r_\theta+s_n}\right)^{\frac{3}{2}} \frac{(r+s_n)^{\frac{1}{2}}(r_\theta+s)^{\frac{1}{2}}}{(r+s)^{\frac{1}{2}}(r_\theta+s_n)^{\frac{1}{2}}} F\left(-\frac{1}{2}, -\frac{1}{2}, 1; p\right)$$

i.e. $$W = (r+s)w ,$$

where $$w = \frac{(r+1)^{\frac{1}{2}}(r_\theta+s)^{\frac{1}{2}}}{(r_\theta+s_n)^{2}} F\left(-\frac{1}{2}, -\frac{1}{2}, 1 ; p\right) .$$

For the formation problem, the procedure is similar to the above with $s_n \equiv 1$.

§E. The derivation and solution of equation (3.5.13)

On multiplying equation (3.5.8) by $(r_\theta+s_n)^{\frac{3}{2}}$ and then using (3.5.12), it follows that

$$M(\epsilon) = 1 - \int_{\sigma_n}^{\epsilon} M(\sigma)\, \frac{(r_\theta+s_n)^{\frac{3}{2}}(r+s)^{-\frac{1}{2}}}{(\frac{3}{2}r - \frac{1}{2}s - \dot{\xi})}\; k_1(\sigma,\epsilon)\frac{dr}{d\sigma}\, d\sigma .$$

From (3.5.11) and the expansions for $(r+s)^{-\frac{1}{2}}$, $(\frac{3}{2}r-\frac{1}{2}s-\dot{\xi})^{-1}$, $(r_\theta+s_n)^{\frac{3}{2}}$ and $\frac{dr}{d\sigma}$ noted in §C, we obtain,

$$\frac{(r_\theta+s_n)^{\frac{3}{2}}(r+s)^{-\frac{1}{2}}}{(\frac{3}{2}r - \frac{1}{2}s - \dot{\xi})}\; k_1(\sigma,\epsilon)\frac{dr}{d\sigma} = \frac{1}{\sigma}\left[1 + \frac{3}{8}\sigma - \frac{3}{64}\sigma^2 - \frac{21}{256}\sigma^3 + \frac{9}{128}\epsilon\,\sigma^2 + O(\sigma^4)\right] .$$

The integral equation (3.5.8) may then be written as

$$M(\epsilon) = 1 - \int_{\sigma_n}^{\epsilon} \frac{M(\sigma)}{\sigma}\left[1 + \frac{3}{8}\sigma - \frac{3}{64}\sigma^2 - \frac{21}{256}\sigma^3 + \frac{9}{128}\epsilon\,\sigma^2 + O(\sigma^4)\right] d\sigma .$$

This equation is equivalent to the ordinary differential equation (3.5.14), that is

$$\text{(i)}\qquad \epsilon^2\frac{d^2M}{d\epsilon^2} + \left[1 + \frac{3}{8}\sigma - \frac{3}{64}\epsilon^2 - \frac{3}{256}\epsilon^3 + O(\epsilon^4)\right]\epsilon\,\frac{dM}{d\epsilon}$$
$$- \left[1 + \frac{3}{64}\epsilon^2 - \frac{15}{128}\epsilon^3 + O(\epsilon^4)\right] M = 0 ,$$

with initial conditions:

$$\text{(ii)}\qquad M(\sigma_n) = 1 ,\quad M'(\sigma_n) = -\frac{1}{\sigma_n}\left[1 + \frac{3}{8}\sigma_n - \frac{3}{64}\sigma_n^2 - \frac{3}{256}\sigma_n^3 + O(\sigma_n^4)\right] .$$

To solve this equation, we may assume a series solution of the form

$$M(\epsilon) = \sum_{r=0}^{\infty} a_r\,\epsilon^{n+r} ,\qquad a_o = 1 .$$

The indicial equation then yields $(n+1)(n-1) = 0$, and on equating to zero the coefficients of ϵ, ϵ^2 and ϵ^3, we obtain

$$a_1 = -\frac{3}{8(n+2)}, \quad a_2 = \frac{9}{64(n+2)(n+3)} + \frac{3}{64(n+3)}$$

and
$$a_3 = \frac{\frac{3n}{256} - \frac{15}{128} - \frac{9}{256}\frac{(n+4)}{(n+3)}}{(n+2)(n+4)} .$$

If A, B are arbitrary constants, then the solution of (i) is

$$M(\epsilon) = A\epsilon\left[1 - \frac{1}{8}\epsilon + \frac{3}{128}\epsilon^2 - \frac{51}{5120}\epsilon^3 + 0(\epsilon^4)\right]$$
$$+ \frac{B}{\epsilon}\left[1 - \frac{3}{8}\epsilon + \frac{3}{32}\epsilon^2 - \frac{5}{64}\epsilon^3 + 0(\epsilon^4)\right] .$$

The constants A, B are determined by conditions (ii), which yield

$$A = \frac{99}{1024}\sigma_n^2\left[1+0(\sigma_n)\right] ,$$
$$B = \sigma_n\left[1 + \frac{3}{8}\sigma_n + \frac{3}{64}\sigma_n^2 - \frac{37}{1024}\sigma_n^3 + 0(\sigma_n^4)\right] .$$

The solution for $M(\epsilon)$ is then

$$M(\epsilon) = \frac{\sigma_n}{\epsilon}\left[1 + \frac{3}{8}(\sigma_n - \epsilon) + \frac{3}{64}(\sigma_n^2 - 3\sigma_n\epsilon + 2\epsilon^2)\right.$$
$$\left. - \frac{1}{1024}\{80(\epsilon^3 - \sigma_n^3) - 99\sigma_n(\epsilon^2 - \sigma_n^2) + 18\sigma_n(\sigma_n^2 + \sigma_n\epsilon - 2\epsilon^2)\} + 0(\sigma_n^4)\right] .$$

From (3.5.12), (3.5.7) it follows that

$$M(\epsilon) = \left(\frac{\frac{3}{2}r_\theta - \frac{1}{2}s_\theta - \dot{\xi}}{\frac{3}{2}r_n - \frac{1}{2}s_n - \dot{\xi}_n}\right)\left(\frac{r_\theta + s_\theta}{r_n + s_n}\right)^{\frac{1}{2}} \frac{t(r_\theta, s_\theta)}{t_n} ,$$

$$= \frac{\epsilon}{\sigma_n}\left[\frac{1 + \frac{1}{24}\epsilon - \frac{1}{32}\epsilon^2 + \frac{5}{512}\epsilon^3 + 0(\epsilon_n^4)}{1 + \frac{1}{24}\sigma_n - \frac{1}{32}\sigma_n^2 + \frac{5}{512}\sigma_n^3 + 0(\sigma_n^4)}\right]\frac{t(\epsilon)}{t_n} .$$

$$\therefore \quad \frac{t(\epsilon)}{t_n} = \frac{\sigma_n^2}{\epsilon^2}\left[1 + \frac{5}{12}(\sigma_n - \epsilon) + \frac{1}{288}(9\sigma_n^2 - 50\sigma_n\epsilon + 41\epsilon^2)\right.$$
$$\left. - \frac{1}{27648}(999\sigma_n^3 + 360\sigma_n^2\epsilon - 4313\sigma_n\epsilon^2 + 2954\epsilon^3) + 0(\sigma_n^4)\right] .$$

§F. Derivation of (3.6.1)

In the 'simple wave' approximation t is determined as a function of r from equation (3.2.1). When the incident simple wave is point-centred at the origin then from (3.2.3)a, $\frac{dF}{dr} = -\frac{3}{2}t_n$ and the initial condition appropriate to (3.2.1) is given by (3.2.3)b.

On changing the parameter in (3.2.1) from r to σ, (3.2.1) reduces to

$$\frac{dt}{d\sigma} + \frac{2}{\sigma[1 - \frac{5}{24}\sigma]} t = 0 \quad ,$$

with the initial condition: $t = t_n$, $\sigma = \sigma_n$.

The appropriate solution to the above equation is

$$\frac{t}{t_n} = \left(\frac{\sigma_n}{\sigma}\right)^2 \left[\frac{1 - \frac{5}{24}\sigma}{1 - \frac{5}{24}\sigma_n}\right]^2 \quad .$$

§G. Derivation of (3. 7. 3)

The equations of the C^+, C^- characteristics of the incident simple wave are respectively,

(i) $$x - X(r) = (\frac{3}{2}r - \frac{1}{2}s_n)\{t - T(r)\} ,$$

(ii) and $$\frac{dx}{dr} = (\frac{1}{2}r - \frac{3}{2}s_n)\frac{dt}{dr} .$$

On substituting for $\frac{dx}{dr}$ from (i) into (ii), the C^- characteristic of the simple wave through the initial point of decay of the bore is determined by the solution of

$$(r+s_n)\frac{dt}{dr} + \frac{3}{2}t = \lambda(r) ,$$

where $$\lambda(r) = \frac{d}{dr}[X(r) + (\frac{3}{2}r - \frac{1}{2}s_n)\, T(r)] ,$$

subject to the condition: $t = t_n$, $r = r_n$.

The required solution is then

$$t = t_n\left(\frac{r_n + s_n}{r + s_n}\right)^{\frac{3}{2}} + (r+s_n)^{-\frac{3}{2}} \int_{r_n}^{r} \lambda(z)\,(z+s_n)^{\frac{1}{2}}\, dz .$$

$$\S\text{H.}\qquad G[r(H), s(H)] = -\left[\frac{H + \sqrt{\frac{1+H}{2}}}{H - \sqrt{\frac{1+H}{2}}}\right]^3 , \quad H \geq 1 .$$

$G(r, s)$ is given as a function of r, s by equation (3.1.10). From (3.1.8), we have

$$\begin{matrix} r \\ s \end{matrix} = \sqrt{H} \pm \frac{H-1}{2}\sqrt{\frac{1+H}{2H}} , \quad H \geq 1 .$$

Hence: $\frac{1}{2}r - \frac{3}{2}s - \frac{(r+s)}{4\sqrt{2}}\sqrt{(r+s)^2 + 4} = -\left(\sqrt{H} + \sqrt{\frac{1+H}{2H}}\right)$,

$$\frac{3}{2}r - \frac{1}{2}s - \frac{(r+s)}{4\sqrt{2}}\sqrt{(r+s)^2 + 4} = \left(\sqrt{H} - \sqrt{\frac{1+H}{2H}}\right),$$

$$(r+s)\left[3(r+s)^2 + 4\right] + 8\sqrt{2}\, s\sqrt{(r+s)^2 + 4} = \frac{8}{\sqrt{H}}\left[2H^2 + H + 1 + 2H\sqrt{2(H+1)}\right] ,$$

$$(r+s)\left[3(r+s)^2 + 4\right] - 8\sqrt{2}\, r\sqrt{(r+s)^2 + 4} = \frac{8}{\sqrt{H}}\left[2H^2 + H + 1 - 2H\sqrt{2(H+1)}\right]$$

$\therefore$ From (3.1.10),

$$G[r(H), s(H)] = -\left[\frac{\sqrt{H} + \sqrt{\frac{1+H}{2H}}}{\sqrt{H} - \sqrt{\frac{1+H}{2H}}}\right]\left[\frac{2H^2 + H + 1 + 2H\sqrt{2(H+1)}}{2H^2 + H + 1 - 2H\sqrt{2(H+1)}}\right]$$

i.e. $$G[r(H), s(H)] = -\left[\frac{\sqrt{H} + \sqrt{\frac{1+H}{2H}}}{\sqrt{H} - \sqrt{\frac{1+H}{2H}}}\right]^3 , \quad H \geq 1 .$$

§ I. The expansions (3.7.20).

In terms of $q = c_n - c$, relations (3.7.19) are:

$$\text{(i)} \quad \dot{\xi} = (c_n - q)\sqrt{\frac{1+c_n^2}{2}} \left[1 + \frac{q^2 - 2c_n q}{1+c_n^2}\right]^{\frac{1}{2}}$$

$$= c_n \sqrt{\frac{1+c_n^2}{2}} \left[1 - \frac{2c_n^2+1}{c_n(c_n^2+1)} q + 0(q^2)\right] ,$$

$$\text{i.e.} \quad \dot{\xi} = \dot{\xi}(r_n, s_n) - \frac{2c_n^2+1}{\sqrt{2(c_n^2+1)}} q + 0(q^2) .$$

$$\text{(ii)} \quad \begin{matrix} r \\ s \end{matrix} = c_n - q \pm \frac{(c_n^2 - 1 - 2c_n q + q^2)}{2\sqrt{2}\, c_n (1 - \frac{q}{c_n})} \sqrt{(1+c_n^2)(1 + \frac{q^2 - 2q\, c_n}{1+c_n^2})}$$

$$= c_n - q \pm \frac{(c_n^2 - 1)\sqrt{1+c_n^2}}{2\sqrt{2}\, c_n} \left[1 + \frac{q^2 - 2c_n q}{c_n^2 - 1}\right] \left[1 - \frac{q}{c_n}\right]^{-1} \left[1 + \frac{q^2 - 2q\, c_n}{1+c_n^2}\right]^{\frac{1}{2}}$$

$$= c_n - q \pm \frac{(c_n^2 - 1)\sqrt{1+c_n^2}}{2\sqrt{2}\, c_n} \left[1 - \frac{2c_n^4 + c_n + 1}{c_n(c_n^4 - 1)} q + \frac{c_n^6 - 4c_n^4 + c_n^2 + 2}{2c_n^2(1+c_n^2)(c_n^4-1)} q^2 + 0(q^3)\right] .$$

$$\therefore \quad \begin{matrix} r \\ s \end{matrix} = \begin{matrix} r_n \\ s_n \end{matrix} - \left[1 \pm \frac{2c_n^4 + c_n^2 + 1}{2\sqrt{2}\, c_n^2 \sqrt{1+c_n^2}} q\right] \pm \frac{2c_n^6 + 3c_n^4 - 3c_n^2 - 2}{4c_n^3 \sqrt{2(1+c_n^2)}\,(1+c_n^2)} q^2 + 0(q^3)$$

$$\text{i.e.} \quad \begin{matrix} r \\ s \end{matrix} = \begin{matrix} r_n \\ s_n \end{matrix} - [1 \pm \alpha_1 q] \pm \alpha_2 q^2 + 0(q^3) .$$

§A. The particular integrals of equations (4.2.11), (4.2.12)

The relevant equations are:

$$(1)\qquad t\frac{\partial \alpha'}{\partial t} + x\frac{\partial \alpha'}{\partial x} + \alpha' - \frac{3-\gamma}{\gamma+1}\beta' = 2c_{sw}\frac{dw}{dy} ,$$

$$(2)\qquad t\frac{\partial \beta'}{\partial t} + \left(\frac{3-\gamma}{\gamma+1}x - 4\frac{\gamma-1}{\gamma+1}\beta_1 t\right)\frac{\partial \beta'}{\partial x} = -2c_{sw}\frac{dw}{dy} .$$

By changing to characteristic variables, it may be seen that the homogeneous forms of the above equations are satisfied by the arbitrary functions

$$\alpha' = \frac{3-\gamma}{\gamma+1} Z^{-\frac{1}{2}} H(Z) + \frac{1}{t} G\left(\frac{x}{t}\right) , \qquad \beta' = 2Z^{\frac{1}{2}}\frac{dH}{dZ} .$$

To derive the particular integrals of the non-homogeneous equations, we proceed as follows. Let us assume that such a solution of (2) may be written in the form $\beta' = f(x,t)g(y)$, where f and g are to be determined. Then on substituting for β' in (2), we obtain

$$t\left[g\frac{\partial f}{\partial t} + f\frac{dg}{dy}\frac{\partial y}{\partial t}\right] + \left[\frac{3-\gamma}{\gamma+1}x - 4\frac{\gamma-1}{\gamma+1}\beta_1 t\right]\left[g\frac{\partial f}{\partial x} + f\frac{dg}{dy}\frac{\partial y}{\partial x}\right] = -2c_{sw}\frac{dw}{dy} .$$

Clearly, if we choose $f(x,t)$ such that $f(x,t) = c_{sw}$, then this equation may be written as

$$(3)\qquad \left[\left(\frac{3-\gamma}{\gamma+1}x - 4\frac{\gamma-1}{\gamma+1}\beta_1 t\right)\frac{\partial y}{\partial x} + t\frac{\partial y}{\partial t}\right]\frac{dg}{dy} + \left[\frac{t}{c_{sw}}\frac{\partial c_{sw}}{\partial t}\right.$$
$$\left. + \left(\frac{3-\gamma}{\gamma+1}x - 4\frac{\gamma-1}{\gamma+1}\beta_1 t\right)\frac{1}{c_{sw}}\frac{\partial c_{sw}}{\partial x}\right] g = -2\frac{dw}{dy} .$$

From (4.2.1), (4.2.5) it follows that

$$\frac{\partial c_{sw}}{\partial t} = -\frac{\gamma-1}{\gamma+1}\frac{x}{t^2} , \quad \frac{\partial c_{sw}}{\partial x} = \frac{\gamma-1}{\gamma+1}\frac{1}{t} , \quad \frac{\partial y}{\partial t} = 2\left(\frac{\gamma-1}{\gamma+1}\right)c_{sw}\left(c_{sw} - \frac{x}{t}\right)t^{-\frac{3-\gamma}{\gamma+1}}$$

and $\dfrac{\partial y}{\partial x} = 2\left(\dfrac{\gamma-1}{\gamma+1}\right)c_{sw}\, t^{-\frac{3-\gamma}{\gamma+1}}$.

Consequently, $[(\frac{3-\gamma}{\gamma+1}x - 4\frac{\gamma-1}{\gamma+1}\beta_1 t)\frac{\partial y}{\partial x} + t\frac{\partial y}{\partial t}] = -2\frac{\gamma-1}{\gamma+1}y$

and $\frac{1}{c_{sw}}[t\frac{\partial c_{sw}}{\partial t} + (\frac{3-\gamma}{\gamma+1}x - 4\frac{\gamma-1}{\gamma+1}\beta_1 t)\frac{\partial c_{sw}}{\partial x}] = -2\frac{\gamma-1}{\gamma+1}$.

Equation (3) then reduces to the simple form

$$\frac{dg}{dy} + \frac{1}{y}g = \frac{\gamma+1}{\gamma-1}\frac{1}{y}\frac{dw}{dy} ,$$

which determines the function g as

$$g = \frac{\gamma+1}{\gamma-1}\frac{w(y)}{y} .$$

The particular integral of equation (2) is then

$$(4) \qquad \beta' = \frac{\gamma+1}{\gamma-1}\frac{c_{sw}}{y}w(y) .$$

To determine the particular integral of equation (1), a similar procedure is used. We assume that α' is of the form $\alpha' = p(x,t)\,q(y)$, where p and q are to be determined. With β' given by (4), equation (1) then becomes

$$t[q\frac{\partial p}{\partial t} + p\frac{dq}{dy}\frac{\partial y}{\partial t}] + x[q\frac{\partial p}{\partial x} + p\frac{dq}{dy}\frac{\partial y}{\partial x}] + pq = \frac{3-\gamma}{\gamma-1}\frac{c_{sw}}{y}w(y) + 2c_{sw}\frac{dw}{dy} .$$

As before, if we now choose $p(x,t) = c_{sw}$, then this equation may be written as

$$(5) \qquad [t\frac{\partial y}{\partial t} + x\frac{\partial y}{\partial x}]\frac{dq}{dy} + [1 + \frac{t}{c_{sw}}\frac{\partial c_{sw}}{\partial t} + \frac{x}{c_{sw}}\frac{\partial c_{sw}}{\partial x}]q = \frac{3-\gamma}{\gamma+1}\frac{w(y)}{y} + 2\frac{dw}{dy} .$$

But, $t\frac{\partial y}{\partial t} + x\frac{\partial y}{\partial x} = 2(\frac{\gamma-1}{\gamma+1})y$, and $t\frac{\partial c_{sw}}{\partial t} + x\frac{\partial c_{sw}}{\partial x} = 0$.

Consequently equation (5) reduces to the simple form

$$\frac{dq}{dy} + \frac{\gamma+1}{2(\gamma-1)y} q = \frac{(3-\gamma)(\gamma+1)}{2(\gamma-1)^2} \frac{w(y)}{y} + \frac{\gamma+1}{(\gamma-1)y} \frac{dw}{dy} ,$$

which then determines q as

$$q = \frac{\gamma+1}{\gamma-1} \frac{w(y)}{y} .$$

The particular integral of (1) is then

$$\alpha' = \frac{\gamma+1}{\gamma-1} \frac{c_{sw}}{y} w(y) .$$

§B. The constants T_1, T_2, T_3 of equations (4.3.4), (4.3.5)

From the Rankine-Hugoniot relations (1.3.5), we may write u, c and S as functions of the parameter $X = \frac{c_o}{\dot{\xi}}$, that is

$$u = \frac{2}{\gamma+1} c_o \left(\frac{1-X^2}{X}\right),$$

$$c = \frac{c_o}{(\gamma+1)X} \sqrt{\{2\gamma-(\gamma-1)X^2\}\{(\gamma-1)+2X^2\}}$$

and $$\frac{S-S_o}{c_v} = \log\{2\gamma-(\gamma-1)X^2\} + \gamma \log\{(\gamma-1)+2X^2\} - 2\log X .$$

On differentiating the above relations w.r.t $\dot{\xi}$ we then obtain

$$\frac{du}{d\dot{\xi}} = \frac{2}{\gamma+1}(1+X^2),$$

$$\frac{dc}{d\dot{\xi}} = \frac{2(\gamma+X^4)}{(\gamma+1)(\gamma-1)\sqrt{\{2\gamma-(\gamma-1)X^2\}\{(\gamma-1)+2X^2\}}}$$

and $$\frac{1}{c_v}\frac{dS}{d\dot{\xi}} = \frac{4\gamma(\gamma-1)X(1+X^2)^2}{c_o[\{2\gamma-(\gamma-1)X^2\}\{(\gamma-1)+2X^2\}]} .$$

The constants T_1, T_2, T_3 are obtained from those relations where X has the value $X = \frac{c_o}{\dot{\xi}(N)}$. Thus

$$T_1 = \left(\frac{d\alpha}{d\dot{\xi}}\right)_n = \frac{1}{\gamma+1}\left[(1+X^2) + \frac{2(\gamma+X^4)}{\sqrt{\{2\gamma-(\gamma-1)X^2\}\{(\gamma-1)+2X^2\}}}\right],$$

$$T_2 = \left(\frac{d\beta}{d\dot{\xi}}\right)_n = \frac{1}{\gamma+1}\left[-(1+X^2) + \frac{2(\gamma+X^4)}{\sqrt{\{2\gamma-(\gamma-1)X^2\}\{(\gamma-1)+2X^2\}}}\right]$$

and

$$T_3 = \left(\frac{dS}{d\xi}\right)_n = \frac{4\gamma(\gamma-1)c_v\, X(1+X^2)^2}{c_o[\{2\gamma-(\gamma-1)X^2\}\{(\gamma-1)+2X^2\}]} .$$

§C. The solution of equation (4. 4. 6)

After substituting for $(c_{sw})_s$, $\frac{d \log Z_s}{d\tau}$ and $\lambda(\tau)$ in terms of τ from (4. 4. 6)b. and (4. 3. 3), equation (4. 4. 6) may be written as

$$[T_1 - T_3\{c_1 - \frac{\gamma-1}{2} k(1 - \frac{\tau_n}{\tau})\}]\ddot{\epsilon}(\tau) + \frac{1}{2}\frac{\gamma+1}{\gamma-1}(T_1 - \frac{3-\gamma}{\gamma+1}T_2)(\frac{\nu}{x_o+\nu\tau} - \frac{3-\gamma}{\gamma+1}\frac{1}{\tau})\dot{\epsilon}(\tau)$$

$$= -\frac{1}{2}\frac{\gamma+1}{\gamma-1}[\frac{\nu}{x_o+\nu\tau} - \frac{3-\gamma}{\gamma+1}\frac{1}{\tau}]k(1 - \frac{\tau_n}{\tau}) - k\frac{\tau_n}{\tau^2} ,$$

where $x_o = x_n - \dot{\xi}(N)t_n$; $\nu = \dot{\xi}(N) - 2\beta_1$; $k = \frac{2}{\gamma+1}[u_1 + c_1 - \dot{\xi}(N)]$.

If a new set of constants, K_i, is defined by

$K_1 = T_1 - T_3(c_1 - \frac{\gamma-1}{2}k)$; $K_2 = \frac{\gamma-1}{2}k\,T_3\,\tau_n$;

$K_3 = \frac{k}{2}(\frac{\gamma+1}{\gamma-1})\tau_n$; $K_4 = \frac{\nu}{2}(\frac{\gamma+1}{\gamma-1})(T_1 - \frac{3-\gamma}{\gamma+1}T_2)$;

$K_5 = \frac{1}{2}(\frac{3-\gamma}{\gamma+1})(T_1 - \frac{3-\gamma}{\gamma+1}T_2)$; $K_6 = \frac{k\nu}{2}(\frac{\gamma+1}{\gamma-1})(1 + \frac{\nu\tau_n}{x_o})$;

$K_7 = \frac{k}{2}\frac{\gamma+1}{\gamma-1}(\frac{3-\gamma}{\gamma+1} + \frac{\nu\tau_n}{x_o}$,

then the above equation, after some manipulation, is of the form

$$(1)\quad \ddot{\epsilon}(\tau) + \left[\frac{\frac{1}{\nu}(\frac{K_4 X_o}{X_o K_1 + K_2\nu})}{\tau + \frac{X_o}{\nu}} + \frac{\frac{1}{K_1}(\frac{K_4 K_2}{X_o K_1 + K_2\nu} - K_5)}{\tau - \frac{K_2}{K_1}}\right]\dot{\epsilon}(\tau)$$

$$= \frac{\frac{1}{K_1}\left[\frac{K_6K_2}{X_oK_1+K_2\nu} + K_7 - \frac{K_3K_1}{K_2}\right]}{\tau - \frac{K_2}{K_1}} + \frac{\frac{1}{\nu}\left(\frac{K_6X_0}{X_oK_1+K_2\nu}\right)}{\tau + \frac{X_o}{\nu}} + \frac{\frac{K_3}{K_2}}{\tau}$$

If we now put

$$b_1 = \frac{X_o}{\nu} ; \quad b_2 = \frac{K_2}{K_1} ; \quad a_1 = \frac{K_4 X_o}{\nu(X_oK_1+K_2\nu)} ;$$

$$a_2 = \frac{1}{K_1}\left(\frac{K_4K_2}{X_oK_1+K_2\nu} - K_5\right) ; \quad a_3 = \frac{1}{K_1}\left(\frac{K_6K_2}{X_oK_1+K_2\nu} + K_7 - \frac{K_3K_1}{K_2}\right) ;$$

$$a_4 = \frac{1}{\nu}\left(\frac{K_6 X_o}{X_oK_1+K_2\nu}\right) ; \quad a_5 = \frac{K_3}{K_2} .$$

then equation (1) is simply

$$\ddot{\epsilon}(\tau) + \left(\frac{a_1}{\tau+b_1} + \frac{a_2}{\tau-b_2}\right)\dot{\epsilon}(\tau) = \frac{a_3}{\tau-b_2} + \frac{a_4}{\tau+b_1} + \frac{a_5}{\tau} ,$$

which then yields

$$(2) \qquad (\tau+b_1)^{a_1}(\tau-b_2)^{a_2}\dot{\epsilon}(\tau) = \int_{\tau_n}^{\tau}\left(\frac{a_3}{v-b_2} + \frac{a_4}{v+b_1} + \frac{a_5}{v}\right)(v+b_1)^{a_1}(v-b_2)^{a_2}\, dv ,$$

since $\dot{\epsilon}(\tau_n) = 0$.

If the variable of integration is changed from v to u , where

$$v = u(\tau - \tau_n) + \tau_n$$

and if parameters w_1 , w_2 , w_3 , r_1 , r_2 , r_3 are defined thus

$$w_1 = \frac{\tau - \tau_n}{\tau_n + b_1} \; ; \qquad r_1 = \frac{a_4}{\tau_n + b_1}$$

$$w_2 = \frac{\tau - \tau_n}{\tau_n - b_2} \; ; \qquad r_2 = \frac{a_3}{\tau_n - b_2} \; , \quad \tau_n \neq b_2$$

$$w_3 = \frac{\tau - \tau_n}{\tau_n} \; ; \qquad r_3 = \frac{a_5}{\tau_n}$$

then relation (2) may be rewritten in the form

$$\dot{\epsilon}(\tau) = (\tau - \tau_n) \left(\frac{\tau_n + b_1}{\tau + b_1}\right)^{a_1} \left(\frac{\tau_n - b_2}{\tau - b_2}\right)^{a_2} \int_0^1 \left(\frac{r_2}{1 + w_2 u} + \frac{r_1}{1 + w_1 u} + \frac{r_3}{1 + w_3 u}\right) (1 + w_1 u)^{a_1} (1 + w_2 u)^{a_2} \, du \; .$$

By examination of the forms for T_1, T_2, T_3 as functions of X given in §B, it may be seen that for $\gamma = 1.4$ $b_2 < \tau_n$ for all X in $0 \leq X \leq 1$. If γ is such that $\tau_n = b_2$, then the solution presented must be modified.

§D. The solution for the shock locus as derived from shock-expansion theory.

The solution detailed here is due entirely to Meyer (1959) and the present author has reproduced it mainly for its elegance and for the fact that the original report may be difficult to obtain.

The governing equations of motion of the system may be written as

$$(1) \qquad \alpha,_t + (u+c)\alpha,_x = \frac{c}{2\gamma(\gamma-1)c_v}\,[S,_t + (u+c)S,_x] \;,$$

$$(2) \qquad \beta,_t + (u-c)\beta,_x = \frac{c}{2\gamma(\gamma-1)c_v}\,[S,_t + (u-c)S,_x] \;,$$

$$(3) \qquad S,_t + u\,S,_x = 0 \;,$$

where a comma denotes differentiation w. r. t the indicated variable.

From (1), (2) it then follows that

$$u,_t + (u+c)u,_x = \frac{2c}{\gamma(\gamma-1)c_v}\,[S,_t + u\,S,_x] - 2[\beta,_t + u\,\beta,_x] \;.$$

On the basis of shock-expansion theory, the Riemann invariant β is entirely a function of the entropy S, that is, $\beta = \beta(S)$ and consequently β is constant on the particle paths of the system. In conjunction with (3), this then implies that the particle velocity u is constant on the C^+ characteristics. Since we can write

$$u,_t + (u+c)\,u,_x + \frac{c}{p}\,[p,_t + (u+c)\,p,_x] = 0 \;,$$

it follows that a similar property holds for the pressure, p. The C^+ characteristics are lines of constant velocity and pressure, even if they are not lines of constant density nor straight as in the 'simple wave' approximation.

The simplification arising from the assumption, $\beta = \beta(S)$, is that it includes (2) as one of its consequences, so that only two characteristic equations remain to be satisfied, that is (2) and (3). A convenient approach is to choose the particle velocity u and the mass ψ as independent variables, where

$$\psi,_x = \rho \;, \qquad \psi,_t = -\rho u \;.$$

ψ, u are characteristic variables since ψ = constant on the particle paths and u = constant on the C^+ characteristics. We can then write

$$p = p(u)\,, \quad S = S(\psi)\,, \quad \beta = \beta(\psi)\,.$$

Since $\phi,_x = \frac{1}{D}\left[t,_u\phi,_\psi - t,_\psi\phi,_u\right]$, $\phi,_t = -\frac{1}{D}\left[x,_u\phi,_\psi - x,_\psi\phi,_u\right]$

where $D = x,_\psi t,_u - x,_u t,_\psi$, and it is assumed that $D \neq 0$,

it follows that equations (1), (3) become respectively

$$\text{(5)} \qquad \begin{aligned} x,_\psi &= (u+c)t,_\psi \\ x,_u &= u\,t,_u \end{aligned} \quad .$$

On the C^+ characteristics,

$$d\psi = \psi,_t + (u+c)\psi,_u = c\rho\,dt\,,$$

$$\text{(6)} \qquad \text{and so } \; t,_\psi = \frac{1}{c\rho} = \frac{d\bar{w}}{dp}\exp\left(\frac{S-S_o}{2\gamma c_v}\right)\,,$$

$$\text{where } \; \bar{w}(p) = \frac{2}{\gamma-1}c_o\left(\frac{p}{p_o}\right)^{\frac{\gamma-1}{2\gamma}}\,.$$

Since the piston path is a possible particle path, it follows that on the piston $\beta = \beta_p$ = constant, so that the relation between pressure and velocity can be written more explicitly in the form

$$\text{(7)} \qquad u = \frac{2c}{\gamma-1} - 2\beta_p = \bar{w}(p)\exp\left(\frac{S_p - S_o}{2\gamma c_v}\right) - 2\beta_p\,.$$

Moreover, on the piston, the piston acceleration b_p is given by

$$(t,_u)^{-1} = b_p\,.$$

Consequently, if the second derivative $t,_{\psi,u}$ is computed from equation (6) with the aid of (7) and then integrated along a C^+ characteristic from the piston path to an arbitrary particle path, the expression

$$\text{(8)} \qquad t,_u = \frac{1}{b_p} + \frac{d}{d\bar{w}}\left(\frac{d\bar{w}}{dp}\right)\int_0^\psi \exp\left(\frac{S(\psi') - S_p}{2\gamma c_v}\right)d\psi'$$

for $t,_u$ as a function of ψ and u is obtained.

On the shock curve, $dx = \dot{\xi}\, dt$, so that the slope of the shock curve $\psi = \psi_s(u)$ in the (ψ, u)-plane is

$$(9) \qquad \frac{d\psi_s}{du} = \frac{\dot{\xi}\, t,_u - x,_u}{x,_\psi - \dot{\xi}\, t,_\psi} = \left(\frac{\dot{\xi} - u}{u+c-\dot{\xi}}\right) c\rho\, t,_u(\psi_s, u)$$

by equation (5).

When the values for $t,_u(\psi_s, u)$ are equated from (8), (9) we then obtain

$$(10) \qquad \left(\frac{u+c-\dot{\xi}}{\dot{\xi}-u}\right) \frac{1}{c\rho} \frac{d\psi_s}{du} = \frac{1}{b_p} + \frac{d}{d\bar{w}}\left(\frac{d\bar{w}}{dp}\right) \int_{\Gamma}^{u} \exp\left(\frac{S(u)-S_p}{2\gamma c_v}\right) \frac{d\psi_s}{du}\, du .$$

Since S is known as a function $S(u)$ on the shock and $c, \rho, \bar{w}$ and $\dot{\xi}$ are similarly known from the shock relations, the above relation represents an ordinary differential equation for the quantity

$$(10a) \qquad f(u) = \int_{\Gamma}^{u} \exp\left(\frac{S(u)-S_p}{2\gamma c_v}\right) \frac{d\psi_s}{du}\, du ,$$

$$(11) \qquad \text{that is,} \quad \frac{df(u)}{du} + \frac{d \log g(u)}{du} f(u) = h(u) ,$$

$$(12) \qquad \text{where } h(u) = \left(\frac{\dot{\xi} - u}{u+c-\dot{\xi}}\right) \frac{c\rho}{b_p}\, e^{\frac{S(u)-S_p}{2\gamma c_v}} .$$

$$\text{and } \frac{d \log g}{du} = - b_p h(u) \frac{d}{d\bar{w}}\left(\frac{d\bar{w}}{dp}\right)$$

$$= \frac{\gamma+1}{2\gamma} \frac{h b_p}{p} ,$$

with $g(\Gamma) = 1$.

Consequently,

$$g(u)\, f(u) = f_o + \int_{\Gamma}^{u} gh\, du .$$

The position x_s and time t_s on the shock are given by

$$\rho_o x_s = \psi_s = \rho_o x_m + \int_{f_o}^{f} \exp\left(\frac{S_p - S(u)}{2\gamma c_v}\right) df ,$$

(13)

$$t_s = t_m + \frac{1}{\rho_o} \int_{f_o}^{f} \frac{1}{\xi} \exp\left(\frac{S_p - S(u)}{2\gamma c_v}\right) df .$$

The constants are $\Gamma = 0$, $S_p = S_o$ and $t_m = \frac{x_m}{c_o} = \frac{2c_o}{(\gamma+1) b_o} = \frac{f_o}{c_o \rho_o}$, when the initial velocity of the piston is zero and its initial acceleration, $b_o > 0$; and $S_p = S(\Gamma)$, $x_m = t_m = 0 = f_o$, when the initial velocity of the piston is $\Gamma > 0$. The solution (12) must be modified if $\Gamma = 0$ and the piston acceleration is initially an increasing function of time so that the shock is formed in the simple wave domain.

§E. The coefficients $\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3$ of equation (6.4.16)

The parametric solution for the path of the shock in the physical plane when derived from shock-expansion theory is given by equation (13) §D. In the present case, the formation of a shock wave, we assume that the motion of the piston is given by

$$(1) \qquad X(t_p) = \frac{1}{2}X^{(2)}t_p^{\,2} + \frac{1}{6}X^{(3)}t_p^{\,3}, \quad X^{(2)} \neq 0 ,$$

following the notation adopted in §4 Chapter VI.

The velocity of the piston is

$$\dot{X}(t_p) = u = X^{(2)}t_p + \frac{1}{2}X^{(3)}t_p^{\,2} ,$$

and since we are interested only in the initial stages of the formation of the shock wave, this relation may be inverted to give t_p as a function of u, that is,

$$(2) \qquad t_p = \frac{u}{X^{(2)}} - \frac{X^{(3)}}{2\{X^{(2)}\}^2}u^2 + \frac{\{X^{(3)}\}^2}{2\{X^{(2)}\}^5}u^3 + O(u^4) .$$

From (1), the acceleration of the piston, b_p, is

$$\ddot{X}(t_p) = b_p = X^{(2)} + X^{(3)}t_p ,$$

and on substituting for t_p from (2), we obtain

$$(3) \qquad b_p = X^{(2)} + \Delta_1 u + \Delta_2 u^2 + \Delta_3 u^3 + O(u^4) ,$$

$$\text{where } \Delta_1 = \frac{X^{(3)}}{X^{(2)}} ; \; \Delta_2 = \frac{\{X^{(3)}\}^2}{2\{X^{(2)}\}^3} ; \; \Delta_3 = \frac{\{X^{(3)}\}^3}{2\{X^{(2)}\}^5} .$$

We further require the following series expansions of the shock relations in the parameter δ, defined in §2 Chapter VI.

$$u = \frac{2k-1}{k}c_o\left[1 - \frac{\delta^2}{2} + \frac{\delta^3}{2} - \frac{\delta^4}{2} + O(\delta^5)\right]$$

$$c = c_o\left[1 + \frac{1}{k}\delta - \frac{1}{2k}\delta^2 + \frac{1}{k}\delta^3 - \frac{5k+2}{4k^2}\delta^4 + O(\delta^5)\right] ,$$

$$(4)\qquad \frac{S-S_o}{c_v} = \frac{2(2k+1)}{3k^2}\,[\delta^3 + 0(\delta^4)]\ ,$$

$$\frac{p}{p_o} = [1 + \frac{2k+1}{k}\,\delta + \frac{2k+1}{2k}\,\delta^2 + 0(\delta^3)]\ ,$$

$$\frac{\rho}{\rho_o} = [1 + \frac{2k-1}{k}\,\delta + \frac{2k^2-5k+2}{2k^2}\,\delta^2 - \frac{2k^2-3k+2}{k^3}\,\delta^3 + 0(\delta^4)]\ ,$$

where $k = \frac{1}{2}\frac{\gamma+1}{\gamma-1}$ and the series converge for $|\delta| < 0.62$ when $\gamma = 1.4$. After some manipulation we obtain the following expansions:

$$(5)\qquad \left(\frac{\dot{\xi}-u}{u+c-\dot{\xi}}\right) c\rho\, \ell^{\frac{S-S_o}{2\gamma c_v}} = \frac{c_o\rho_o}{\delta}\,[1 + \frac{2k+1}{k}\delta + \frac{k+1}{k}\delta^2 + \frac{(2k-1)(4-3k)}{12k^2}\,\delta^3 + 0(\delta^4)]\ ,$$

$$(6)\qquad b_p = X^{(2)}\,[1 + \bar{\Delta}_1\delta + \bar{\Delta}_2\delta^2 + \bar{\Delta}_3\delta^3 + 0(\delta^4)]\ ,$$

where $X^{(2)}\bar{\Delta}_1 = \frac{2k-1}{k}c_o\,\Delta_1$; $X^{(2)}\bar{\Delta}_2 = (\frac{2k-1}{k}c_o)^2\Delta_2 - \frac{1}{2}(\frac{2k-1}{k}c_o)\Delta_1$;

$$X^{(2)}\bar{\Delta}_3 = (\frac{2k-1}{k}c_o)^3\Delta_3 - (\frac{2k-1}{k}c_o)^2\Delta_2 + \frac{1}{2}(\frac{2k-1}{k}c_o)\,\Delta_1\ .$$

Consequently, from equations (4), (5), (6) and relations (12) §D, we obtain the required expansions, in the parameter δ, for the functions $h(u)$, $\frac{1}{g}\frac{dg}{du}$, that is:

$$(7)\qquad h(u) = H(\delta) = \frac{c_o\rho_o}{X^{(2)}\delta}\,[1 + A_1\delta + A_2\delta^2 + A_3\delta^3 + 0(\delta^4)]\ ,$$

where $A_1 = \frac{2k+1}{k} - \bar{\Delta}_1$; $A_2 = \frac{k+1}{k} + \bar{\Delta}_1^{\,2} - \Delta_2 - \frac{2k+1}{k}\bar{\Delta}_1$;

$$A_3 = \frac{(2k-1)(4-3k)}{12k^2} + (2\bar{\Delta}_1\bar{\Delta}_2 - \bar{\Delta}_1^{\,3} - \bar{\Delta}_3) + \frac{2k+1}{k}(\bar{\Delta}_1^{\,2} - \bar{\Delta}_2) - \frac{k+1}{k}\bar{\Delta}_1$$

$$(8)\qquad \frac{1}{g}\frac{dg}{du} = \frac{2k}{2k+1}\;\frac{b_p h}{p} = \frac{2k}{2k+1}\;\frac{c_o\rho_o}{p_o\delta}\,[1 + B_2\delta^2 + B_3\delta^3 + 0(\delta^4)]\ ,$$

where $B_2 = \bar{\Delta}_2 + A_2 + \bar{\Delta}_1 A_1 - \frac{2k+1}{2k}$;

$$B_3 = \bar{\Delta}_3 + A_3 + \bar{\Delta}_1 A_2 + \bar{\Delta}_2 A_1 - \frac{2k+1}{k}(\bar{\Delta}_2 + A_2 + \bar{\Delta}_1 A_1) + \frac{(2k+1)^2}{2k^2} .$$

The function $f(u) = F(\delta)$, defined by (10a) of §D, satisfies equation (11) of §D. On substituting the above expansions for $h(u)$, $\frac{1}{g}\frac{dg}{du}$ into this equation we obtain, after some manipulation, the result

$$(9) \quad \frac{dF}{d\delta} + \frac{2}{\delta}[1-\delta+C_2\delta^2+C_3\delta^3+0(\delta^4)]F = \frac{2k-1}{k}\,\frac{c_o^2 \rho_o}{X^{(2)}}\,\frac{1}{\delta}[1+D_1\delta+D_2\delta^2+D_3\delta^3+0(\delta^4)] ,$$

where $C_2 = \frac{3}{2} + B_2$; $C_3 = B_3 - B_2 - 2$;

$$D_1 = A_1 - 1 ; \quad D_2 = A_2 - A_1 + \frac{3}{2} ; \quad D_3 = A_3 - A_2 + \frac{3}{2}A_1 - 2 .$$

This equation is to be solved subject to the initial condition, $f = f_o$, $\delta = 0$, and the appropriate solution is

$$(10) \quad F(\delta) = f_o[1 + F_1\delta + F_2\delta^2 + F_3\delta^3 + 0(\delta^4)] ,$$

where $F_1 = \frac{2}{3}[D_1+1]$; $F_2 = \frac{1}{2}(D_2-C_2) + \frac{1}{3}(D_1+1)$;

$$F_3 = \frac{2}{5}[D_3-2D_2+2D_1+D_1+D_1C_2+\frac{2}{3}C_3-2C_2-\frac{4}{3}] + [\frac{4}{3}-2C_2-\frac{2}{3}C_3]$$
$$+ \frac{2}{3}(D_1-2)(2-C_2) + (D_2-2D_1+C_2+2) .$$

From equations (12) of §D we obtain the required series expansions for x , t in terms of δ :

$$x = x_m[1 + F_1\delta + F_2\delta^2 + F_3\delta^3 + 0(\delta^4)] ,$$

$$t = t_m[1 + F_1\delta + (F_2 - \frac{F_1}{2})\delta + (F_3 - \frac{2}{3}F_2 + \frac{F_1}{3})\delta^3 + 0(\delta^4)] .$$

By back-substituting for the various coefficients, F_1, F_2, F_3 may be obtained as functions of the quantities c_o, k, $X^{(2)}$, $X^{(3)}$.

$$F_1 = \frac{2}{3}\left[\frac{3k+1}{k} - \left(\frac{2k-1}{k}c_o\right)\frac{X^{(3)}}{\{X^{(2)}\}^2}\right] ,$$

$$F_2 = \left[\frac{2k+1}{12k} - \frac{7k+6}{12k}\left(\frac{2k-1}{k}c_o\right)\frac{X^{(3)}}{\{X^{(2)}\}^2} + \frac{3}{4}\left(\frac{2k-1}{k}c_o\right)^2\frac{\{X^{(3)}\}^2}{\{X^{(2)}\}^4}\right] ,$$

$$F_3 = \left[\frac{(2k+1)(k+2)}{30k^2} - \frac{k-4}{30k}\left(\frac{2k-1}{k}c_o\right)\frac{X^{(3)}}{\{X^{(2)}\}^2} + \frac{3k+6}{10k}\left(\frac{2k-1}{k}c_o\right)^2\frac{\{X^{(3)}\}^2}{\{X^2\}^4}\right.$$

$$\left. - \left(\frac{2k-1}{k}c_o\right)^3\frac{\{X^{(3)}\}^3}{\{X^{(2)}\}^6}\right] .$$

In terms of the parameter $\pi = \left(\frac{2k-1}{k}c_o\right)\frac{X^{(3)}}{\{X^{(2)}\}^2}$; F_1, F_2 and F_3 are given by

$$F_1 = \frac{2}{3}\left(\frac{2k+1}{k} - \pi\right)$$

$$F_2 = \frac{2k+1}{12k} - \frac{7k+6}{12k}\pi + \frac{3}{4}\pi^2 ,$$

$$F_3 = \frac{(2k+1)(k+2)}{30k^2} - \frac{k-4}{30k}\pi + \frac{3k+6}{10k}\pi^2 - \pi^3 .$$

The coefficients quoted in (6.4.16) are then given by

$$\bar{\mu}_1 = F_1 ; \quad \bar{\mu}_2 = F_2 - \frac{F_1}{2} ; \quad \bar{\mu}_3 = F_3 - \frac{2}{3}F_2 + \frac{F_1}{3} ,$$

and hence,

$$\bar{\mu}_1 = \frac{2}{3}\left(\frac{2k+1}{k} - \pi\right) ,$$

$$\bar{\mu}_2 = -\frac{1}{4}\left(\frac{2k+1}{k} + \frac{k+2}{k}\pi - 3\pi^2\right) ,$$

$$\bar{\mu}_3 = \frac{(2k+1)(3k+1)}{15k^2} + \frac{4k+13}{30k}\pi + \frac{3-k}{5k}\pi^2 - \pi^3 .$$

It is worthwhile to note that in deriving the above expressions, series expansions of the Rankine-Hugoniot relations were employed which were taken to terms of order four in δ . In the solution to this problem by the method of Chapter VI those expansions required to be taken only to terms of order three in δ .

§F. Relations (6.5.7).

Relations (6.5.7) are:

$$(1)\quad \left(\frac{d^2x}{dt^2}\right)_n = -\frac{c_o\delta_n}{2t_n}\left[1 - \frac{k-1}{2k}\delta_n^{\ 2} + 0(\delta_n^{\ 3})\right] ,$$

$$(2)\quad \left(\frac{d^3x}{dt^3}\right)_n = \frac{3c_o\delta_n}{4t_n^{\ 2}}\left[1 - \frac{k-1}{k}\delta_n^{\ 2} + 0(\delta_n^{\ 3})\right] .$$

In the case of the decay of a shock wave by a point-centred simple rarefaction wave the velocity of the piston is discontinuous at the origin of the co-ordinate system and the solution for the path of the shock wave as represented by (13) §D must be modified. Any particle path of the incident simple wave can be chosen as a piston path which results in the same motion of the modified shock wave. As a convenient piston path, we choose the particle path of the simple wave which passes through the initial point of decay of the shock wave. The equation of this curve is represented by

$$\frac{t_p}{t_n} = \left(\frac{u + 2\beta_1}{u_1 + 2\beta_1}\right)^{-\frac{\gamma+1}{\gamma-1}} = \left(\frac{c}{c_1}\right)^{-2k} , \quad k = \frac{1}{2}\left(\frac{\gamma+1}{\gamma-1}\right) ,$$

where t_p denotes the measure of time along the particle path.

The acceleration of the piston, b_p, is then given by

$$\frac{1}{b_p} = -\frac{2k}{2k-1}\frac{t_n}{c_1}\left(\frac{c}{c_1}\right)^{-(2k+1)}$$

If the series expansion for c in terms of δ is now substituted into the above relation, then we obtain

$$(3)\quad \frac{1}{b_p} = -\frac{2k}{2k-1}\frac{t_n}{c_o}\frac{c_1}{c_o}^{-2k}\left[1 - \frac{2k+1}{k}\delta + \frac{(2k+1)(3k+2)}{2k^2}\delta^{\ 2} + 0(\delta^3)\right] .$$

Only quadratic expansions of the type indicated above in the square brackets require to be examined in order to derive relations (1), (2) above. This is not to say that third order terms in the series expansions in δ of the shock relations are ignored. The third order terms require to be considered when an expansion whose leading term is δ is obtained. In such a case, the factor δ is removed outside the bracket and the resulting polynomial in δ is considered only to terms of order two. Such a case arises when the following expansion is derived.

$$(4) \qquad \left(\frac{\dot{\xi}-u}{u+c-\dot{\xi}}\right) c\rho\, \ell^{\frac{S(n)-S_p}{2\gamma c_v}} = c_o\rho_o \frac{1}{\delta}\left[1+\frac{2k+1}{k}\delta+\frac{k+1}{k}\delta^2+0(\delta^3)\right] .$$

From (3), (4) it then follows that

$$(5) \qquad h(u) = H(\delta) = A_1 \frac{1}{\delta}\left[1+\frac{\delta^2}{2k}+0(\delta^3)\right] ,$$

$$\text{where } A_1 = -\frac{2k}{2k-1} t_n\rho_o \left(\frac{c_1}{c_o}\right)^{2k} \ell^{-\frac{S_p-S_o}{2\gamma c_v}} = \frac{2k}{2k-1} t_n\rho_o \left[1+2\delta_n+\frac{k-1}{k}\delta_n^2+0(\delta_n^3)\right] .$$

By definition,

$$\frac{1}{g}\frac{dg}{du} = \frac{\gamma+1}{2\gamma}\frac{b_p h}{p} = \frac{2k}{2k+1}\frac{c_o\rho_o}{p_o} \ell^{\frac{S_p-S_o}{2\gamma c_v}} \frac{1}{\delta}\left[1+\frac{1}{2k}\delta^2+0(\delta^3)\right] .$$

$$\therefore \qquad \frac{1}{G}\frac{dG}{d\delta} = \frac{2}{\delta}\left[1-\delta+\frac{3k+1}{2k}\delta^2+0(\delta^3)\right], \quad \text{where } g(u) = G(\delta) ,$$

$$(6) \quad \therefore \quad G(\delta) = A_2\left(\frac{\delta}{\delta_n}\right)^2 \left[1-2\delta+\frac{7k+1}{2k}\delta^2+0(\delta^3)\right] ,$$

$$\text{where } A_2 = \left[1+2\delta_n+\frac{k-1}{2k}\delta_n^2+0(\delta_n^3)\right] .$$

It is easily verified that $g(\delta_n) = 1$, as is required.

From (5), (6) we then obtain

$$H(\delta)\, G(\delta)\frac{du}{d\delta} = \frac{2k-1}{k} A_1 A_2 c_o \frac{\delta}{\delta_n^2}\left[1-3\delta+\frac{7k+1}{k}\delta^2+0(\delta^3)\right] .$$

$$(7)\quad \therefore \int_{\delta_n}^{\delta} H(\delta)\, G(\delta) \frac{du}{d\delta}\, d\delta = F(\delta)\, G(\delta)\ , \qquad \text{where } f(u) = F(\delta)\ ,$$

$$= \frac{2k-1}{2k} A_1 A_2 c_o \left[\left(\frac{\delta}{\delta_n}\right)^2 \left(1 - 2\delta + \frac{7k+1}{2k}\delta^2 + 0(\delta^3)\right) - \left(1 - 2\delta_n + \frac{7k+1}{2k}\delta_n^{\,2} + 0(\delta_n^{\,3})\right)\right] .$$

From (6), (7) we then find, after some algebra, that

$$(8)\qquad F(\delta) = \frac{2k-1}{2k} A_1 c_o \left[1 - b_1 \ell(\delta)\right] ,$$

where $b_1 = 1 - 2\delta_n + \frac{7k+1}{2k}\delta_n^{\,2} + 0(\delta_n^{\,3})$,

$$\ell(\delta) = \left(\frac{\delta_n}{\delta}\right)^2 \left[1 + 2\delta + \frac{k-1}{2k}\delta^2 + 0(\delta^3)\right] .$$

From equation (13) $^{\delta}$D, we obtain

$$\frac{dx_s}{d\delta} = \frac{1}{\rho_o} \ell^{\frac{S_p - S_n}{2\gamma c_v}} \frac{dF(\delta)}{d\delta}$$

which, on using relations (5) and (8), reduces to

$$(9)\qquad \frac{dx_s}{d\delta} = \nu \frac{d\ell}{d\delta} = -2\nu \frac{\delta_n^{\,2}}{\delta^3}\left[1 + \delta + 0(\delta^3)\right] ,$$

where $\nu = c_o t_n \left(\frac{c_1}{c_o}\right)^{2k} b_1 = c_o t_n \left[1 + \frac{k-1}{2k}\delta_n^{\,2} + 0(\delta_n^{\,3})\right]$.

On the shock wave, $dx_s = c_o(1+\delta)dt_s$, and consequently from (9),

$$\frac{d}{dt_s} = \frac{c_o(1+\delta)}{\nu\left(\frac{d\ell}{d\delta}\right)} \frac{d}{d\delta} .$$

Thus,

$$\frac{d^2x_s}{dt_s^2} = \frac{c_o^{\,2}(1+\delta)}{\nu\,(\frac{d\ell}{d\delta})} = -\,\frac{c_o^{\,2}\,\delta^3}{2\nu\;\delta_n^{\,2}}\;[1+0(\delta^3)]\;,$$

$$\frac{d^3x_s}{dt_s^{\,3}} = \frac{c_o^{\,3}(1+\delta)}{\nu^2(\frac{d\ell}{d\delta})}\;[\frac{d}{d\delta}\{\frac{1+\delta}{\frac{d\ell}{d\delta}}\}\,] = \frac{3\,c_o^{\,3}\,\delta^5}{4\,\nu^2\,\delta_n^{\,4}}\,[1+0(\delta^3)]\;.$$

Hence at the initial point of decay of the shock wave, the deceleration and rate of change of deceleration of the shock wave are given by the expressions,

$$\left(\frac{d^2x_s}{dt_s^{\,2}}\right)_n = -\,\frac{c_o^{\,2}}{2\nu}\,\delta_n\,[1+0(\delta_n^{\,3})] = -\,\frac{c_o\,\delta_n}{2t_n}\,[1-\frac{k-1}{2k}\,\delta_n^{\,2}+0(\delta_n^{\,3})]\;,$$

$$\left(\frac{d^3x_s}{dt_s^{\,3}}\right)_n = \frac{3\,c_o^{\,3}}{4\nu^2}\,\delta_n\,[1+0(\delta_n^{\,3})] = \frac{3\,c_o\,\delta_n}{4\,t_n^{\,2}}\,[1-\frac{k-1}{k}\,\delta_n^{\,2}+0(\delta_n^{\,3})]\;.$$